

THE UNIVERSAL ASSOCIATIVE ENVELOPE OF THE ANTI-JORDAN TRIPLE SYSTEM OF $n \times n$ MATRICES

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ABSTRACT. We show that the universal associative enveloping algebra of the simple anti-Jordan triple system of all $n \times n$ matrices ($n \geq 2$) over an algebraically closed field of characteristic 0 is finite dimensional. We investigate the structure of the universal envelope and focus on the monomial basis, the structure constants, and the center. We explicitly determine the decomposition of the universal envelope into matrix algebras. We classify all finite dimensional irreducible representations of the simple anti-Jordan triple system, and show that the universal envelope is semisimple. We also provide an example to show that the universal enveloping algebras of anti-Jordan triple systems are not necessary to be finite-dimensional.

1. INTRODUCTION

Anti-Jordan triple systems were introduced by Faulkner and Ferrar in [8]. The classification of finite-dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic 0 was given by Bashir [1, Theorem 6].

Definition 1.1. [1] A vector space V over a field F of characteristic $\neq 2$ endowed with a trilinear operation $V \times V \times V \rightarrow V$, $(a, b, c) \rightarrow \langle abc \rangle$ is said to be an *anti-Jordan triple system* if the following conditions are fulfilled for all $a, b, c, d, e \in V$:

$$\langle abc \rangle = -\langle cba \rangle, \quad \langle ab \langle cde \rangle \rangle = \langle \langle abc \rangle de \rangle + \langle c \langle bad \rangle e \rangle + \langle cd \langle abe \rangle \rangle.$$

If A is an associative algebra, A defines an anti-Jordan triple system A_- relative to the product $\langle abc \rangle = abc - cba$.

Definition 1.2. A *representation* of an anti-Jordan triple system \mathfrak{J} is a homomorphism $\rho: \mathfrak{J} \rightarrow (\text{End } V)_-$ from \mathfrak{J} to the anti-Jordan triple system of endomorphisms of a vector space V . In other words, ρ is a linear mapping that satisfies

$$\rho(\langle abc \rangle) = \rho(a)\rho(b)\rho(c) - \rho(c)\rho(b)\rho(a),$$

for all $a, b, c \in \mathfrak{J}$. Two representations ρ_1 and ρ_2 of an anti-Jordan triple system \mathfrak{J} on the same vector space V are *equivalent* if there exists an invertible endomorphism T such that $\rho_2(a) = T^{-1}\rho_1(a)T$ for all $a \in \mathfrak{J}$.

In this paper we use the theory of non-commutative Gröbner bases to prove that the universal enveloping algebra of the simple anti-Jordan triple system of all $n \times n$ matrices is finite-dimensional. This theory was used by Bergman [3] to

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give a new proof of the PBW theorem and was used recently by Elgendy[5] and Elgendy and Bremner[7] to construct universal associative envelopes of nonassociative triple systems and universal envelopes of the $(n+1)$ -dimensional n -Lie algebras respectively.

This paper is structured as follows. In Section 2, we recall basic results on non-commutative Gröbner bases. In Section 3, we prove that the universal enveloping algebra of the simple anti-Jordan triple system of $n \times n$ matrices over an algebraically closed field is finite-dimensional using Gröbner bases in free associative algebras. In Section 4, we determine the structure constants of the universal enveloping algebra. In Section 5, we determine the center of the universal enveloping algebra. In the last section, we explicitly determine the complete decomposition of the universal enveloping algebra into a direct sum of matrix algebras. We also provide an example of a non-simple anti-Jordan triple system with infinite dimensional envelope. For examples of simple anti-Jordan triple systems with infinite dimensional envelopes see [6, 9].

Unless otherwise stated, we assume throughout that all vector spaces are over an algebraically closed field F of characteristic 0.

2. PRELIMINARIES

In this section we recall the basic definitions and results in the theory of non-commutative Gröbner bases in free associative algebras following [4, 7].

Definition 2.1. Let $X = \{x_1, \dots, x_n\}$ be a set of symbols with the total order $x_i < x_j$ if and only if $i < j$. The *free monoid* generated by X is the set X^* of all (possibly empty) words $w = x_{i_1} \cdots x_{i_k}$ ($k \geq 0$) with the (associative) operation of concatenation. For $w = x_{i_1} \cdots x_{i_k} \in X^*$ the *degree* is $\deg(w) = k$. The *degree-lexicographical* (*deglex*) order $<$ on X^* is defined as follows: $u < v$ if and only if either (i) $\deg(u) < \deg(v)$ or (ii) $\deg(u) = \deg(v)$ and $u = wx_iu'$, $v = wx_jv'$ where $x_i < x_j$ ($w, u', v' \in X^*$). The *free (unital) associative algebra* generated by X is the vector space $F\langle X \rangle$ with basis X^* and multiplication extended bilinearly from concatenation in X^* .

Definition 2.2. The *support* of a noncommutative polynomial $f \in F\langle X \rangle$ is the set of all monomials $w \in X^*$ that occur in f with nonzero coefficient. The *leading monomial* of $f \in F\langle X \rangle$, denoted $\text{LM}(f)$, is the highest element of the support of f with respect to deglex order. If I is any ideal of $F\langle X \rangle$ then the set of *normal words* modulo I is defined by $N(I) = \{u \in X^* \mid u \neq \text{LM}(f) \text{ for any } f \in I\}$. We write $C(I)$ for the subspace of $F\langle X \rangle$ spanned by $N(I)$.

Proposition 2.3. If $I \subset F\langle X \rangle$ is an ideal then $F\langle X \rangle = C(I) \oplus I$.

Definition 2.4. Let $G \subset F\langle X \rangle$ be a subset generating an ideal $I \subset F\langle X \rangle$. A noncommutative polynomial $f \in F\langle X \rangle$ is in *normal form modulo G* if no monomial occurring in f has a factor of the form $\text{LM}(g)$ for any $g \in G$. A subset $G \subset I$ is a *Gröbner basis* of I if for all $f \in I$ there is a $g \in G$ such that $\text{LM}(g)$ is a factor of $\text{LM}(f)$. A subset $G \subset F\langle X \rangle$ is *self-reduced* if every $g \in G$ is in normal form modulo $G \setminus \{g\}$ and every $g \in G$ is *monic*: the coefficient of $\text{LM}(g)$ is 1.

Definition 2.5. Let $g, h \in F\langle X \rangle$ be two monic noncommutative polynomials. Assume that $\text{LM}(g)$ is not a factor of $\text{LM}(h)$ and that $\text{LM}(h)$ is not a factor of $\text{LM}(g)$. Let $u, v \in X^*$ be such that $\text{LM}(g)u = v\text{LM}(h)$, u is a proper right

factor of $\text{LM}(h)$, and v is a proper left factor of $\text{LM}(g)$. In this case the element $gu - vh \in F\langle X \rangle$ is called a *composition* of g and h .

Theorem 2.6. *If $I \subset F\langle X \rangle$ is an ideal generated by a self-reduced set G , then G is a Gröbner basis of I if and only if for all compositions f of the elements of G the normal form of f modulo G is zero.*

3. THE UNIVERSAL ASSOCIATIVE ENVELOPING ALGEBRA

Let \mathfrak{J} be the anti-Jordan triple system of all $n \times n$ matrices over an algebraically closed field F of characteristic 0 with triple product $\langle a, b, c \rangle = abc - cba$.

Definition 3.1. Let $\Omega = \{1, 2, \dots, n\}$ be a finite index set. Let $B = \{E_{i,j}\}_{i,j \in \Omega}$ be an ordered basis of \mathfrak{J} , where $E_{i,j}$ is the matrix with a single 1, in the i th row and j th column, and zeros elsewhere. The structure constants for \mathfrak{J} are

$$\langle E_{i,j}, E_{k,\ell}, E_{m,t} \rangle = \delta_{j,k} \delta_{\ell,m} E_{i,t} - \delta_{t,k} \delta_{\ell,i} E_{m,j}, \quad \text{for all } i, j, k, \ell, m, t \in \Omega.$$

Consider the bijection $\phi: B \rightarrow X = \{e_{i,j}\}_{i,j \in \Omega}$ defined by $\phi(E_{i,j}) = e_{i,j}$. We extend ϕ to a linear map $\phi: \mathfrak{J} \rightarrow F\langle X \rangle$. Throughout this paper we use the deglex order $<$ where $e_{i,j} < e_{k,\ell}$ if either $i < k$, or $i = k$ and $j < \ell$.

Definition 3.2. Let $G \subset F\langle X \rangle$ consist of these elements $(i, j, k, r, s, t \in \Omega)$:

$$\begin{aligned} \mathcal{R}_1^{(i,j,k,t)} &= e_{i,j} e_{j,k} e_{k,t} - e_{k,t} e_{j,k} e_{i,j} - e_{i,t} \quad (k < i), \\ \mathcal{R}_2^{(i,j,t)} &= e_{i,j} e_{j,i} e_{i,t} - e_{i,t} e_{j,i} e_{i,j} - e_{i,t} \quad (t < j), \\ \mathcal{R}_3^{(i,j,k,t)} &= e_{i,j} e_{k,i} e_{t,k} - e_{t,k} e_{k,i} e_{i,j} + e_{t,j} \quad (t < i), \\ \mathcal{R}_4^{(i,j,k)} &= e_{i,j} e_{k,i} e_{i,k} - e_{i,k} e_{k,i} e_{i,j} + e_{i,j} \quad (k < j), \\ \mathcal{R}_5^{(i,j,k,t,r,s)} &= e_{i,j} e_{k,t} e_{r,s} - e_{r,s} e_{k,t} e_{i,j} \quad (r < i, j \neq k \text{ or } t \neq r, s \neq k \text{ or } t \neq i), \\ \mathcal{R}_6^{(i,j,k,t,s)} &= e_{i,j} e_{k,t} e_{i,s} - e_{i,s} e_{k,t} e_{i,j} \quad (s < j, j \neq k \text{ or } t \neq i, s \neq k \text{ or } t \neq i). \end{aligned}$$

Let $I \subset F\langle X \rangle$ be the ideal generated by G . We write $\mathfrak{A} = F\langle X \rangle / I$ with surjection $\pi: F\langle X \rangle \rightarrow \mathfrak{A}$ sending f to $f + I$, and $i = \pi \circ \phi$ for the natural map $i: \mathfrak{J} \rightarrow \mathfrak{A}$.

Lemma 3.3. *The unital associative algebra \mathfrak{A} and the linear map i form the universal associative envelope of the anti-Jordan triple system \mathfrak{J} .*

Our goal in the rest of this section is to derive a Gröbner basis for the ideal I from the set G of generators. This will be achieved by repeatedly calculating normal forms of compositions of generators.

Definition 3.4. We write $\delta_{i,j}$ for the Kronecker delta, and $\widehat{\delta}_{i,j} = 1 - \delta_{i,j}$.

Lemma 3.5. *The set of all normal forms modulo G of nontrivial compositions among elements of G includes the set G_1 which consists of the elements:*

$$\begin{aligned} \mathcal{G}_1^{(r,t,m)} &= e_{r,t} e_{t,m} - e_{r,1} e_{1,m} \quad (m \neq r, t \neq 1), \\ \mathcal{G}_2^{(i,t,\ell)} &= e_{i,t} e_{\ell,i} - e_{1,t} e_{\ell,1} \quad (t \neq \ell, i \neq 1), \\ \mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j} e_{k,\ell} \quad (i \neq \ell, j \neq k). \end{aligned}$$

Proof. For all $s < t$, we consider the following composition:

$$S = \mathcal{R}_2^{(r,t,s)} e_{s,m} - e_{r,t} \mathcal{R}_1^{(t,r,s,m)}.$$

We eliminate from S all occurrences of the leading monomials of G as factors in the monomials; we write \equiv to indicate congruence modulo G :

$$\begin{aligned} S &= -e_{r,s} e_{t,r} e_{r,t} e_{s,m} - e_{r,s} e_{s,m} + e_{r,t} e_{s,m} e_{r,s} e_{t,r} + e_{r,t} e_{t,m} \\ &\equiv -e_{r,s} (e_{s,m} e_{r,t} e_{t,r} - \delta_{m,r} e_{s,r}) - e_{r,s} e_{s,m} + (e_{r,s} e_{s,m} e_{r,t} - \delta_{m,r} e_{r,t}) e_{t,r} + e_{r,t} e_{t,m} \\ &= \delta_{m,r} e_{r,s} e_{s,r} - e_{r,s} e_{s,m} - \delta_{m,r} e_{r,t} e_{t,r} + e_{r,t} e_{t,m}, \end{aligned}$$

using the relations $\mathcal{R}_3^{(t,r,r,s)}$, $\mathcal{R}_5^{(t,r,r,t,s,m)}$, $\mathcal{R}_4^{(r,t,s)}$ and $\mathcal{R}_6^{(r,t,s,m,s)}$. Clearly, if $m = r$ then $S \equiv 0$. Assume $m \neq r$ and obtain the set L of nonzero normal forms of S modulo G :

$$L = \{ \mathcal{N}^{(r,t,m,s)} = e_{r,t} e_{t,m} - e_{r,s} e_{s,m} \mid \text{for all } s < t, m \neq r \}.$$

The set L is not self-reduced. Therefore, for all $1 < s < t \leq n$, we eliminate from the element $\mathcal{N}^{(r,t,m,s)}$ occurrence of the leading monomial of $\mathcal{N}^{(r,s,m,1)}$ and obtain a self-reduced set consisting of the elements $\mathcal{G}_1^{(r,t,m)}$.

For all $(r, \ell) < (i, k)$, we consider the following composition:

$$S_1 = \mathcal{R}_1^{(i,k,r,t)} e_{\ell,s} - e_{i,k} \mathcal{R}_5^{(k,r,r,t,\ell,s)} \quad (t \neq \ell, \text{ and } s \neq r \text{ or } t \neq k).$$

We eliminate from S_1 all occurrences of the leading monomials of elements of G :

$$\begin{aligned} S_1 &= -e_{r,t} e_{k,r} e_{i,k} e_{\ell,s} - e_{i,t} e_{\ell,s} + e_{i,k} e_{\ell,s} e_{r,t} e_{k,r} \\ &\equiv -e_{r,t} (e_{\ell,s} e_{i,k} e_{k,r} - \delta_{s,i} e_{\ell,r}) - e_{i,t} e_{\ell,s} + e_{r,t} e_{\ell,s} e_{i,k} e_{k,r} \\ &= \delta_{s,i} e_{r,t} e_{\ell,r} - e_{i,t} e_{\ell,s}, \end{aligned}$$

using the relations $\mathcal{R}_3^{(k,r,i,\ell)}$, $\mathcal{R}_5^{(k,r,i,k,\ell,s)}$ and $\mathcal{R}_5^{(i,k,\ell,s,r,t)}$. Hence, for all $(r, \ell) < (i, k)$, the possible (monic) normal forms of S_1 are

$$(1) \quad e_{i,t} e_{\ell,s} \text{ (if } i \neq s), \quad e_{i,t} e_{\ell,i} - e_{r,t} e_{\ell,r} \text{ (if } i = s).$$

For all $(r, t) < (i, k)$, we consider the following composition:

$$\begin{aligned} S_2 &= \mathcal{R}_5^{(i,j,k,\ell,r,s)} e_{t,m} - e_{i,j} \mathcal{R}_5^{(k,\ell,r,s,t,m)} \\ &\quad (s \neq k \text{ or } \ell \neq i, j \neq k \text{ or } \ell \neq r, m \neq r \text{ or } s \neq k) \text{ and } (\ell \neq r \text{ or } s \neq t). \end{aligned}$$

We eliminate from S_2 all occurrences of the leading monomials of elements of G :

$$\begin{aligned} S_2 &= -e_{r,s} e_{k,\ell} e_{i,j} e_{t,m} + e_{i,j} e_{t,m} e_{r,s} e_{k,\ell} \\ &\equiv -e_{r,s} (e_{t,m} e_{i,j} e_{k,\ell} + \delta_{\ell,i} \delta_{j,t} e_{k,m} - \delta_{j,k} \delta_{m,i} e_{t,\ell}) \\ &\quad + (e_{r,s} e_{t,m} e_{i,j} + \delta_{j,t} \delta_{m,r} e_{i,s} - \delta_{s,t} \delta_{m,i} e_{r,j}) e_{k,\ell} \\ &= \delta_{j,t} (-\delta_{\ell,i} e_{r,s} e_{k,m} + \delta_{m,r} e_{i,s} e_{k,\ell}) - \delta_{m,i} (-\delta_{j,k} e_{r,s} e_{t,\ell} + \delta_{s,t} e_{r,j} e_{k,\ell}), \end{aligned}$$

using the relations $\mathcal{R}_1^{(k,\ell,j,m)}$, $\mathcal{R}_3^{(k,\ell,m,t)}$, $\mathcal{R}_5^{(k,\ell,i,j,t,m)}$, $\mathcal{R}_1^{(i,j,r,s)}$, $\mathcal{R}_3^{(i,j,t,r)}$ and $\mathcal{R}_5^{(i,j,t,m,r,s)}$. We first note that if $(m, j, s) = (i, k, t)$ then the (monic) normal form of S_2 modulo G coincides with the element $\mathcal{N}^{(r,k,\ell,t)}$, so we ignore this case. For all $(r, t) < (i, k)$, the possible non-zero (monic) normal forms of S_2 modulo G are

$$(2) \quad \begin{aligned} &e_{r,s} e_{k,m} \quad (m \neq r, s \neq k), & e_{i,s} e_{k,\ell} \quad (\ell \neq i, s \neq k), \\ &e_{r,s} e_{t,\ell} \quad (r \neq \ell, s \neq t), & e_{r,j} e_{k,\ell} \quad (\ell \neq r, j \neq k), \\ &e_{r,t} e_{k,i} \quad (t \neq k, r < i), & e_{i,s} e_{k,i} - e_{r,s} e_{k,r} \quad (s \neq k). \end{aligned}$$

Combining (1) and (2) gives all the possible normal forms of S_1 and S_2 :

$$\begin{aligned}\mathcal{L}^{(i,s,k,r)} &= e_{i,s}e_{k,i} - e_{r,s}e_{k,r} \quad (r < i, s \neq k), \\ \mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j}e_{k,\ell} \quad (i \neq k, j \neq \ell).\end{aligned}$$

We observe that the set $\{\mathcal{L}^{(i,s,k,r)} \mid \text{for all } r < i, s \neq k\}$ is not self-reduced. Therefore, for all $1 < r < i \leq n$, we eliminate from the element $\mathcal{L}^{(i,s,k,r)}$ occurrence of the leading monomial of $\mathcal{L}^{(r,s,k,1)}$ and obtain a self-reduced set consisting of the elements $\mathcal{G}_2^{(i,s,k)}$. For $n = 2$, we cannot obtain $\mathcal{G}_3^{(1,2,1,2)}$, $\mathcal{G}_3^{(2,2,1,1)}$ and $\mathcal{G}_3^{(2,1,2,1)}$ from S_1 or S_2 . Thus, we consider three more compositions in this case:

$$\begin{aligned}S_3 &= \mathcal{R}_6^{(1,2,1,2,1)}e_{1,1} - e_{1,2}\mathcal{R}_4^{(1,2,1)}, & S_4 &= \mathcal{R}_1^{(2,2,1,2)}e_{1,1} - e_{2,2}\mathcal{R}_3^{(2,1,1,1)}, \\ S_5 &= \mathcal{R}_2^{(2,2,1)}e_{2,1} - e_{2,2}\mathcal{R}_6^{(2,2,2,1,1)}.\end{aligned}$$

We eliminate from S_3 all the leading monomials of elements of G and obtain

$$\begin{aligned}S_3 &= -e_{1,1}e_{1,2}^2e_{1,1} + e_{1,2}e_{1,1}^2e_{1,2} - e_{1,2}^2 \\ &\equiv -e_{1,1}(e_{1,1}e_{1,2}^2) + (e_{1,1}^2e_{1,2} - e_{1,2})e_{1,2} - e_{1,2}e_{1,2} = -2e_{1,2}^2,\end{aligned}$$

using the relations $\mathcal{R}_6^{(1,2,1,2,1)}$ and $\mathcal{R}_4^{(1,2,1)}$. Similarly, we can show that $S_4 \equiv -2e_{2,2}e_{1,1}$ and $S_5 \equiv -2e_{2,1}e_{2,1}$. The monic forms of the last three elements give the required elements. This completes the proof. \square

Lemma 3.6. *The set of all normal forms modulo $G \cup G_1$ of nontrivial compositions among elements of $G \cup G_1$ includes the set G_2 which consists of the elements:*

$$\mathcal{G}_4^{(r,i)} = e_{r,i}e_{i,r} - e_{r,1}e_{1,r} + e_{1,1}^2 - e_{1,i}e_{i,1} \quad (r, i \in \Omega \setminus \{1\}).$$

Proof. For all $(s, t) < (r, i)$, we consider the following composition:

$$S = \mathcal{R}_1^{(r,i,s,t)}e_{t,m} - e_{r,i}\mathcal{R}_1^{(i,s,t,m)}.$$

We eliminate from S all occurrences of the leading monomials of elements of G :

$$\begin{aligned}S &= -e_{s,t}e_{i,s}e_{r,i}e_{t,m} - e_{r,t}e_{t,m} + e_{r,i}e_{t,m}e_{s,t}e_{i,s} + e_{r,i}e_{i,m} \\ &\equiv -e_{s,t}(e_{t,m}e_{r,i}e_{i,s} - \delta_{m,r}e_{t,s}) - e_{r,t}e_{t,m} + (e_{s,t}e_{t,m}e_{r,i} - \delta_{m,r}e_{s,i})e_{i,s} + e_{r,i}e_{i,m} \\ &= \delta_{m,r}e_{s,t}e_{t,s} - e_{r,t}e_{t,m} - \delta_{m,r}e_{s,i}e_{i,s} + e_{r,i}e_{i,m},\end{aligned}$$

using the relations $\mathcal{R}_3^{(i,s,r,t)}$, $\mathcal{R}_5^{(i,s,r,i,t,m)}$, $\mathcal{R}_3^{(r,i,t,s)}$ and $\mathcal{R}_5^{(r,i,t,m,s,t)}$. We now eliminate from S all occurrences of the leading monomials of elements of G_1 . Clearly, if $m \neq r$ then $S \equiv 0 \pmod{G_1}$, using the relations $\mathcal{G}_1^{(r,t,m)}$ (if $t \neq 1$) and $\mathcal{G}_1^{(r,i,m)}$. Assume $m = r$ and obtain the set \mathcal{N} of nonzero normal forms of S modulo $G \cup G_1$:

$$\mathcal{N} = \{\mathcal{N}^{(r,i,t,s)} = e_{r,i}e_{i,r} - e_{r,t}e_{t,r} - e_{s,i}e_{i,s} + e_{s,t}e_{t,s} \mid \text{for all } (s, t) < (r, i)\}.$$

We observe that the set \mathcal{N} is not self-reduced and the element $\mathcal{N}^{(r,i,1,1)}$ coincides with $\mathcal{G}_4^{(r,i)}$ for all $r, i \neq 1$. Assume now that $s, t \neq 1$. For all $(s, t) < (r, i)$, we eliminate from $\mathcal{N}^{(r,i,t,s)}$ occurrence of the leading monomials of $\mathcal{N}^{(r,t,1,1)}$, $\mathcal{N}^{(s,i,1,1)}$ and $\mathcal{N}^{(s,t,1,1)}$ and again obtain $\mathcal{G}_4^{(r,i)}$. A similar argument can be used if $s \neq 1$ or $t \neq 1$. The result is a self-reduced set consisting of the elements $\mathcal{G}_4^{(r,i)}$. \square

Lemma 3.7. *The set of all normal forms modulo $G \cup G_1 \cup G_2$ of nontrivial compositions among elements of $G \cup G_1 \cup G_2$ includes the set G_3 which consists of the elements:*

$$\mathcal{G}_5^{(r,i)} = e_{r,1}e_{1,i}e_{i,1} - e_{1,1}^2e_{r,1} - e_{r,1} \quad (r < i; i, r \in \Omega \setminus \{1\}),$$

$$\begin{aligned}
\mathcal{G}_6^{(i,r)} &= e_{i,1}e_{1,i}e_{r,1} - e_{1,1}^2e_{r,1} \quad (i < r; i, r \in \Omega \setminus \{1\}), \\
\mathcal{G}_7^{(t,\ell)} &= e_{1,t}e_{t,1}e_{1,\ell} - e_{1,1}^2e_{1,\ell} \quad (t < \ell; t, \ell \in \Omega \setminus \{1\}), \\
\mathcal{G}_8^{(\ell,t)} &= e_{1,\ell}e_{t,1}e_{1,t} - e_{1,1}^2e_{1,\ell} + e_{1,\ell} \quad (\ell < t; \ell, t \in \Omega \setminus \{1\}), \\
\mathcal{G}_9^{(r)} &= e_{r,1}e_{1,r}e_{r,1} - 2e_{1,1}^2e_{r,1} - e_{r,1} \quad (r \in \Omega \setminus \{1\}), \\
\mathcal{G}_{10}^{(r)} &= e_{1,r}e_{r,1}e_{1,r} - 2e_{1,1}^2e_{1,r} + e_{1,r} \quad (r \in \Omega \setminus \{1\}), \\
\mathcal{G}_{11}^{(r,i,\ell)} &= e_{r,1}e_{1,i}e_{\ell,1} \quad (r \neq i \neq \ell), \\
\mathcal{G}_{12}^{(\ell,i,r)} &= e_{1,\ell}e_{i,1}e_{1,r} \quad (r \neq i \neq \ell), \\
\mathcal{G}_{13}^{(i)} &= e_{1,1}e_{1,i}e_{i,1} - \frac{1}{2}e_{1,1}^3 - \frac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}).
\end{aligned}$$

Proof. For all $r, t, i, \ell, k \in \Omega$, we consider the following six compositions:

$$\begin{aligned}
S_1 &= \mathcal{G}_1^{(r,t,i)}e_{i,\ell} - e_{r,t}\mathcal{G}_1^{(t,i,\ell)} \quad (1 \neq i \neq r, \ell \neq t \neq 1), \\
S_2 &= \mathcal{G}_4^{(r,i)}e_{r,t} - e_{r,i}\mathcal{G}_1^{(i,r,t)} \quad (i, r \neq 1, i \neq t), \\
S_3 &= \mathcal{G}_4^{(i,r)}e_{t,r} - e_{i,r}\mathcal{G}_2^{(r,i,t)} \quad (i, r \neq 1, i \neq t), \\
S_4 &= \mathcal{G}_1^{(r,t,i)}e_{\ell,k} - e_{r,t}\mathcal{G}_3^{(t,i,\ell,k)} \quad (r \neq i \neq \ell, k \neq t \neq 1), \\
S_5 &= \mathcal{G}_2^{(t,\ell,i)}e_{k,r} - e_{t,\ell}\mathcal{G}_3^{(i,t,k,r)} \quad (\ell \neq i \neq r, k \neq t \neq 1), \\
S_6 &= \mathcal{G}_4^{(r,i)}e_{t,\ell} - e_{r,i}\mathcal{G}_3^{(i,r,t,\ell)} \quad (i, r \neq 1, \ell \neq i, r \neq t).
\end{aligned}$$

We eliminate from these compositions all occurrences of the leading monomials of $G \cup G_1 \cup G_2$. For the composition S_1 , we have

$$\begin{aligned}
S_1 &= -e_{r,1}e_{1,i}e_{i,\ell} + e_{r,t}e_{t,1}e_{1,\ell} \\
&\equiv -\delta_{\ell,1}e_{r,1}e_{1,i}e_{i,1} - \widehat{\delta}_{\ell,1}e_{r,1}e_{1,1}e_{1,\ell} + \delta_{r,1}e_{1,t}e_{t,1}e_{1,\ell} + \widehat{\delta}_{r,1}e_{r,1}e_{1,1}e_{1,\ell} \pmod{G_1},
\end{aligned}$$

using the relations $\mathcal{G}_1^{(1,i,\ell)}$ and $\mathcal{G}_1^{(r,t,1)}$. We note first that if $\ell, r \neq 1$ then $S \equiv 0 \pmod{G_1}$. Three cases need to be considered. Case I. If $(\ell, r) = (1, 1)$ then

$$\begin{aligned}
S_1 &\equiv -e_{1,1}e_{1,i}e_{i,1} + e_{1,t}e_{t,1}e_{1,1} \pmod{G_1} \\
&\equiv -e_{1,1}e_{1,i}e_{i,1} + e_{1,1}e_{t,1}e_{1,t} + e_{1,1} \pmod{G},
\end{aligned}$$

using the relation $\mathcal{R}_2^{(1,t,1)}$, since by definition $t \neq 1$. Hence the (monic) normal form of S_1 in this case is

$$(3) \quad \mathcal{G}'^{(t,i)} = e_{1,1}e_{t,1}e_{1,t} - e_{1,1}e_{1,i}e_{i,1} + e_{1,1} \quad (t, i \in \Omega \setminus \{1\}).$$

Case II. If $\ell = 1$ and $r \neq 1$ then

$$\begin{aligned}
S_1 &\equiv -e_{r,1}e_{1,i}e_{i,1} + e_{r,1}e_{1,1}^2 \pmod{G_1} \\
&\equiv -e_{r,1}e_{1,i}e_{i,1} + e_{1,1}^2e_{r,1} + e_{r,1} \pmod{G},
\end{aligned}$$

using the relation $\mathcal{R}_1^{(r,1,1)}$. Clearly, if $r < i$ then the monic form of the last equation coincides with $\mathcal{G}_5^{(r,i)}$. If $i < r$ then the element $e_{r,1}e_{1,i}e_{i,1}$ of the last

equation can be reduced further modulo G : $e_{r,1}e_{1,i}e_{i,1} \equiv e_{i,1}e_{1,i}e_{r,1} + e_{r,1} \pmod{G}$. Using this in the last equation gives $\mathcal{G}_6^{(i,r)}$. Case III. If $\ell \neq 1$ and $r = 1$ then

$$S_1 \equiv -e_{1,1}^2 e_{1,\ell} + e_{1,t} e_{t,1} e_{1,\ell} \pmod{G_1}.$$

Clearly, if $t < \ell$ then the normal form of S_1 in this case coincides with $\mathcal{G}_7^{(t,\ell)}$. If $\ell < t$ then the element $e_{1,t} e_{t,1} e_{1,\ell}$ of the last equation can be reduced further modulo G : $e_{1,t} e_{t,1} e_{1,\ell} \equiv e_{1,\ell} e_{t,1} e_{1,t} + e_{1,\ell} \pmod{G}$. Using this in the last equation gives $\mathcal{G}_8^{(\ell,t)}$.

For the composition S_2 , we have

$$\begin{aligned} S_2 &= -e_{r,1}e_{1,r}e_{r,t} + e_{1,1}^2 e_{r,t} - e_{1,i}e_{i,1}e_{r,t} + e_{r,i}e_{i,1}e_{1,t} \\ &\equiv -\delta_{t,1}e_{r,1}e_{1,r}e_{r,1} - \widehat{\delta}_{t,1}e_{r,1}e_{1,1}e_{1,t} + \delta_{t,1}e_{1,1}^2 e_{r,1} + e_{r,1}e_{1,1}e_{1,t} \pmod{G_1} \\ &\equiv -\delta_{t,1}e_{r,1}e_{1,r}e_{r,1} + \delta_{t,1}(2e_{1,1}^2 e_{r,1} + e_{r,1}) \pmod{G}, \end{aligned}$$

using the relations $\mathcal{G}_1^{(1,r,t)}$, $\mathcal{G}_3^{(1,1,r,t)}$, $\mathcal{G}_3^{(i,1,r,t)}$, $\mathcal{G}_1^{(r,i,1)}$ and $\mathcal{R}_1^{(r,1,1,t)}$. Hence, for $t = 1$ the (monic) normal form of S_2 coincides with $\mathcal{G}_9^{(r)}$. For the composition S_3 , we have

$$\begin{aligned} S_3 &= -e_{i,1}e_{1,i}e_{t,r} + e_{1,1}^2 e_{t,r} - e_{1,r}e_{r,1}e_{t,r} + e_{i,r}e_{1,i}e_{t,1} \\ &\equiv \delta_{t,1}e_{1,1}^2 e_{1,r} - \widehat{\delta}_{t,1}e_{1,r}e_{1,1}e_{t,1} - \delta_{t,1}e_{1,r}e_{r,1}e_{1,r} + e_{1,r}e_{1,1}e_{t,1} \pmod{G_1} \\ &\equiv \delta_{t,1}(e_{1,1}^2 e_{1,r} - e_{1,r}e_{r,1}e_{1,r} + e_{1,1}^2 e_{1,r} - e_{1,r}) \pmod{G}, \end{aligned}$$

using the relations $\mathcal{G}_3^{(1,i,t,r)}$, $\mathcal{G}_3^{(1,1,t,r)}$, $\mathcal{G}_2^{(i,r,1)}$, $\mathcal{G}_2^{(r,1,t)}$ and $\mathcal{R}_4^{(1,r,1)}$. Hence, for $t = 1$ the (monic) normal form of S_3 coincides with $\mathcal{G}_{10}^{(r)}$. Next, we consider the composition S_4 :

$$S_4 = -e_{r,1}e_{1,i}e_{\ell,k} \equiv -\delta_{k,1}e_{r,1}e_{1,i}e_{\ell,1} \pmod{G_1},$$

using the relation $\mathcal{G}_3^{(1,i,\ell,k)}$. Obviously, for $k = 1$ the (monic) normal form of S_4 coincides with $\mathcal{G}_{11}^{(r,i,\ell)}$. Similarly, we can show that for $k = 1$, the (monic) non zero normal form of S_5 coincides with $\mathcal{G}_{12}^{(\ell,i,r)}$. Finally, for the composition S_6 , we have

$$\begin{aligned} S_6 &= -e_{r,1}e_{1,r}e_{t,\ell} + e_{1,1}^2 e_{t,\ell} - e_{1,i}e_{i,1}e_{t,\ell} \\ &\equiv -\delta_{\ell,1}e_{r,1}e_{1,r}e_{t,1} + \delta_{t,1}e_{1,1}^2 e_{1,\ell} + \widehat{\delta}_{t,1}\delta_{\ell,1}e_{1,1}^2 e_{t,1} - \delta_{t,1}e_{1,i}e_{i,1}e_{1,\ell} \pmod{G_1}, \end{aligned}$$

using the relations $\mathcal{G}_3^{(1,r,t,\ell)}$, $\mathcal{G}_3^{(1,1,t,\ell)}$ and $\mathcal{G}_3^{(i,1,t,\ell)}$. Clearly, if $\ell = 1$ and $t \neq 1$ then the (monic) normal form of S_6 coincides with $\mathcal{G}_6^{(r,t)}$ (if $r < t$) and $\mathcal{G}_5^{(t,r)}$ (if $t < r$). If $\ell \neq 1$ and $t = 1$ then the (monic) normal form of S_6 coincides with $\mathcal{G}_7^{(i,\ell)}$ (if $i < \ell$) and $\mathcal{G}_8^{(\ell,i)}$ (if $\ell < i$). If $(\ell, t) = (1, 1)$, since by definition $i, r \neq 1$, we have

$$\begin{aligned} S_6 &\equiv -e_{r,1}e_{1,r}e_{1,1} + e_{1,1}^3 - e_{1,i}e_{i,1}e_{1,1} \pmod{G_1} \\ &\equiv -(e_{1,1}e_{1,r}e_{r,1} - e_{1,1}) + e_{1,1}^3 - (e_{1,1}e_{i,1}e_{1,i} + e_{1,1}) \pmod{G}, \end{aligned}$$

using the relations $\mathcal{R}_3^{(r,1,1,1)}$ and $\mathcal{R}_2^{(1,i,1)}$. Hence, the (monic) normal form of S_6 in this case is

$$(4) \quad \mathcal{G}''^{(i,r)} = e_{1,1}e_{i,1}e_{1,i} + e_{1,1}e_{1,r}e_{r,1} - e_{1,1}^3 \quad (i, r \in \Omega \setminus \{1\}).$$

We note that the set $\mathcal{N} = \{\mathcal{G}'^{(t,i)}, \mathcal{G}''^{(i,r)} \mid \text{for all } i, t, r \in \Omega \setminus \{1\}\}$ of the normal forms (3) and (4) is not self-reduced. So, we eliminate from $\mathcal{G}'^{(i,i)}$ the leading monomial of $\mathcal{G}''^{(i,i)}$ and obtain

$$\mathcal{G}'^{(i,i)} = -2e_{1,1}e_{1,i}e_{i,1} + e_{1,1}^3 + e_{1,1},$$

whose monic form coincides with $\mathcal{G}_{13}^{(i)}$. We now eliminate from $\mathcal{G}''^{(i,i)}$ the leading monomial of $\mathcal{G}_{13}^{(i)}$ and obtain

$$\mathcal{G}''^{(i,i)} = e_{1,1}e_{i,1}e_{1,i} + \frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} - e_{1,1}^3,$$

which coincides with $\mathcal{G}_{14}^{(i)}$. \square

Lemma 3.8. *The set of all normal forms modulo $G \cup G_1 \cup G_2 \cup G_3$ of nontrivial compositions among elements of $G \cup G_1 \cup G_2 \cup G_3$ includes the set G_4 which consists of the elements:*

$$\begin{aligned} \mathcal{G}_{17}^{(i)} &= e_{1,1}^3 e_{1,i} - e_{1,1} e_{1,i}, & \mathcal{G}_{18}^{(i)} &= e_{1,1}^3 e_{i,1} + e_{1,1} e_{i,1} & (i \in \Omega \setminus \{1\}), \\ \mathcal{G}_{19} &= e_{1,1}^5 - e_{1,1}. \end{aligned}$$

Proof. For all $i \in \Omega \setminus \{1\}$ we consider the following three compositions:

$$\begin{aligned} S_1 &= \mathcal{G}_{14}^{(i)} e_{1,i} - e_{1,1} e_{i,1} \mathcal{G}_3^{(1,i,1,i)}, & S_2 &= \mathcal{G}_{13}^{(i)} e_{i,1} - e_{1,1} e_{1,i} \mathcal{G}_3^{(i,1,i,1)}, \\ S_3 &= \mathcal{G}_{13}^{(i)} e_{1,i} e_{i,1} - e_{1,1} e_{1,i} \mathcal{G}_9^{(i)}. \end{aligned}$$

We note that $S_1 = -\frac{1}{2}e_{1,1}^3 e_{1,i} + \frac{1}{2}e_{1,1} e_{1,i}$ and $S_2 = -\frac{1}{2}e_{1,1}^3 e_{i,1} - \frac{1}{2}e_{1,1} e_{i,1}$ are in normal form modulo $G \cup G_1 \cup G_2 \cup G_3$ and the monic forms of S_1 and S_2 coincide with $\mathcal{G}_{17}^{(i)}$ and $\mathcal{G}_{18}^{(i)}$ respectively. For the composition S_3 , we have

$$\begin{aligned} S_3 &= -\frac{1}{2}e_{1,1}^3 e_{1,i} e_{i,1} - \frac{1}{2}e_{1,1} e_{1,i} e_{i,1} + 2e_{1,1} e_{1,i} e_{1,1}^2 e_{i,1} + e_{1,1} e_{1,i} e_{i,1} \\ &\equiv -\frac{1}{2}e_{1,1}^2 \left(\frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} \right) + \frac{1}{2} \left(\frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} \right) \pmod{G_3} = -\frac{1}{4}e_{1,1}^5 + \frac{1}{4}e_{1,1}, \end{aligned}$$

using the relations $\mathcal{G}_{13}^{(i)}$ and $\mathcal{G}_{11}^{(1,i,1)}$, whose monic form coincides with \mathcal{G}_{19} . \square

Lemma 3.9. *The self-reduced form \mathfrak{G} of the set $G \cup G_1 \cup G_2 \cup G_3 \cup G_4$ consists of the elements:*

$$\begin{aligned} \mathcal{G}_0^{(i,j)} &= e_{i,1} e_{1,1} e_{1,j} - e_{1,j} e_{1,1} e_{i,1} - e_{i,j} & (i, j \in \Omega \setminus \{1\}), \\ \mathcal{G}_1^{(i,j,k)} &= e_{i,j} e_{j,k} - e_{i,1} e_{1,k} & (i, j, k \in \Omega; k \neq i, j \neq 1), \\ \mathcal{G}_2^{(i,j,k)} &= e_{i,j} e_{k,i} - e_{1,j} e_{k,1} & (i, j, k \in \Omega; j \neq k, i \neq 1), \\ \mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j} e_{k,\ell} & (i, j, k, \ell \in \Omega; i \neq \ell, j \neq k), \\ \mathcal{G}_4^{(i,j)} &= e_{i,j} e_{j,i} - e_{i,1} e_{1,i} - e_{1,j} e_{j,1} + e_{1,1}^2 & (i, j \in \Omega \setminus \{1\}), \\ \mathcal{G}_5^{(i,j)} &= e_{i,1} e_{1,j} e_{j,1} - e_{1,1}^2 e_{i,1} - e_{i,1} & (i, j \in \Omega; i \neq 1, j \neq i), \\ \mathcal{G}_6^{(i,j)} &= e_{j,1} e_{1,j} e_{i,1} - e_{1,1}^2 e_{i,1} & (i, j \in \Omega \setminus \{1\}; i \neq j), \\ \mathcal{G}_7^{(i,j)} &= e_{1,i} e_{i,1} e_{1,j} - e_{1,1}^2 e_{1,j} & (i, j \in \Omega \setminus \{1\}; i \neq j), \\ \mathcal{G}_8^{(i,j)} &= e_{1,i} e_{j,1} e_{1,j} - e_{1,1}^2 e_{1,i} + e_{1,i} & (i, j \in \Omega; i \neq j, i \neq 1), \\ \mathcal{G}_9^{(i)} &= e_{i,1} e_{1,i} e_{i,1} - 2e_{1,1}^2 e_{i,1} - e_{i,1} & (i \in \Omega \setminus \{1\}), \\ \mathcal{G}_{10}^{(j)} &= e_{1,j} e_{j,1} e_{1,j} - 2e_{1,1}^2 e_{1,j} + e_{1,j} & (j \in \Omega \setminus \{1\}), \\ \mathcal{G}_{11}^{(i,j,k)} &= e_{i,1} e_{1,j} e_{k,1} & (k, i, j \in \Omega; k, i \neq j), \\ \mathcal{G}_{12}^{(i,j,k)} &= e_{1,i} e_{j,1} e_{1,k} & (k, i, j \in \Omega; i, k \neq j), \\ \mathcal{G}_{13}^{(i)} &= e_{1,1} e_{1,i} e_{i,1} - \frac{1}{2}e_{1,1}^3 - \frac{1}{2}e_{1,1} & (i \in \Omega \setminus \{1\}), \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{15}^{(i)} &= e_{1,i}e_{i,1}e_{1,1} - \frac{1}{2}e_{1,1}^3 - \frac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{16}^{(i)} &= e_{i,1}e_{1,i}e_{1,1} - \frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{17}^{(i)} &= e_{1,1}^3e_{1,i} - e_{1,1}e_{1,i} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{18}^{(i)} &= e_{1,1}^3e_{i,1} + e_{1,1}e_{i,1} \quad (i \in \Omega \setminus \{1\}), \\
\mathcal{G}_{19} &= e_{1,1}^5 - e_{1,1}.
\end{aligned}$$

Proof. To obtain the self-reduced set \mathfrak{G} , we need to eliminate from $G \cup \bigcup_{i=1}^4 G_i$ all occurrences of any element of $\left\{ \text{LM}(u) : u \in G \cup \bigcup_{i=1}^4 G_i \right\}$ as a subword of any element of $G \cup \bigcup_{i=1}^4 G_i$. We first note that any element $g \in \bigcup_{i=1}^4 G_i$ is in normal form modulo $G \cup \bigcup_{i=1}^4 G_i \setminus \{g\}$. So we only consider elements of G (see Definition 3.2). For all $k < i$, we have

$$\begin{aligned}
\mathcal{R}_1^{(i,j,k,t)} &= e_{i,j}e_{j,k}e_{k,t} - e_{k,t}e_{j,k}e_{i,j} - e_{i,t} \equiv e_{i,1}e_{1,k}e_{k,t} - e_{k,t}e_{1,k}e_{i,1} - e_{i,t} \pmod{G_1} \\
&\equiv \delta_{t,1} (e_{i,1}e_{1,k}e_{k,1} - e_{1,1}^2e_{i,1}) + \widehat{\delta}_{t,1} (e_{i,1}e_{1,1}e_{1,t} - e_{1,t}e_{1,1}e_{i,1}) - e_{i,t} \pmod{G_3 \cup G_1},
\end{aligned}$$

using the relations $\mathcal{G}_1^{(i,j,k)}$, $\mathcal{G}_2^{(j,k,i)}$, $\mathcal{G}_6^{(k,i)}$, $\mathcal{G}_1^{(1,k,t)}$ and $\mathcal{G}_2^{(k,t,1)}$. For $t \neq 1$ the last result coincides with $\mathcal{G}_0^{(i,t)}$. For $t = 1$, we combine the result with the set $\left\{ \mathcal{G}_5^{(i,k)} \mid \text{for all } 1 < i < k \right\} \subset G_3$ and obtain the set $\left\{ \mathcal{G}_5^{(i,k)} \mid \text{for all } k \neq i \neq 1 \right\}$. For all $t < j$, we have

$$\begin{aligned}
\mathcal{R}_2^{(i,j,t)} &= e_{i,j}e_{j,i}e_{i,t} - e_{i,t}e_{j,i}e_{i,j} - e_{i,t} \equiv e_{i,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{i,j} - e_{i,t} \pmod{G_1} \\
&\equiv \delta_{i,1} (e_{1,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{1,j}) + \widehat{\delta}_{i,1} (e_{i,1}e_{1,1}e_{1,t} - e_{1,t}e_{1,1}e_{i,1}) - e_{i,t} \pmod{G_1},
\end{aligned}$$

using the relations $\mathcal{G}_1^{(j,i,t)}$, $\mathcal{G}_2^{(i,t,j)}$, $\mathcal{G}_1^{(i,j,1)}$ and $\mathcal{G}_2^{(j,1,i)}$. For $i \neq 1$ the last result coincides with $\mathcal{G}_0^{(i,t)}$ (if $t \neq 1$) and $\mathcal{G}_5^{(i,1)}$ (if $t = 1$). For $i = 1$, using the relations $\mathcal{G}_{14}^{(i)}$ and $\mathcal{G}_8^{(t,j)}$, we have

$$\begin{aligned}
\mathcal{R}_2^{(1,j,t)} &\equiv \left[\delta_{t,1} (e_{1,j}e_{j,1}e_{1,1} - e_{1,1}e_{j,1}e_{1,j}) + \widehat{\delta}_{t,1} (e_{1,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{1,j}) - e_{1,t} \right] \\
&\pmod{G_1} \equiv \delta_{t,1} (e_{1,j}e_{j,1}e_{1,1} - \frac{1}{2}e_{1,1}^3 - \frac{1}{2}e_{1,1}) + \widehat{\delta}_{t,1} (e_{1,j}e_{j,1}e_{1,t} - e_{1,1}^2e_{1,t}) \pmod{G_3}.
\end{aligned}$$

Clearly, for $t = 1$ the normal form of $\mathcal{R}_2^{(1,j,t)}$ coincides with $\mathcal{G}_{15}^{(i)}$. For $t \neq 1$, we combine the last result with the set $\left\{ \mathcal{G}_7^{(j,t)} \mid \text{for all } 1 < j < t \right\} \subset G_3$ and obtain the set $\left\{ \mathcal{G}_7^{(j,t)} \mid \text{for all } 1 \neq j \neq t \neq 1 \right\}$. For all $t < i$, we have

$$\begin{aligned}
\mathcal{R}_3^{(i,j,k,t)} &= e_{i,j}e_{k,i}e_{t,k} - e_{t,k}e_{k,i}e_{i,j} + e_{t,j} \equiv e_{i,j}e_{1,i}e_{t,1} - e_{t,1}e_{1,i}e_{i,j} + e_{t,j} \pmod{G_1} \\
&\equiv \delta_{j,1} (e_{i,1}e_{1,i}e_{t,1} - e_{t,1}e_{1,i}e_{i,1}) + \widehat{\delta}_{j,1} (e_{1,j}e_{1,1}e_{t,1} - e_{t,1}e_{1,1}e_{1,j}) + e_{t,j} \pmod{G_1},
\end{aligned}$$

using the relations $\mathcal{G}_2^{(k,i,t)}$, $\mathcal{G}_1^{(t,k,i)}$, $\mathcal{G}_2^{(i,j,1)}$ and $\mathcal{G}_1^{(1,i,j)}$. For $j \neq 1$ the monic form of the last result coincides with $\mathcal{G}_0^{(j,t)}$ (if $t \neq 1$) and $\mathcal{G}_8^{(j,1)}$ (if $t = 1$). For $j = 1$, using the relations $\mathcal{G}_{13}^{(i)}$ and $\mathcal{G}_5^{(t,i)}$, we have

$$\mathcal{R}_3^{(i,1,k,t)} \equiv \delta_{t,1} (e_{i,1}e_{1,i}e_{1,1} - \frac{1}{2}e_{1,1}^3 + \frac{1}{2}e_{1,1}) + \widehat{\delta}_{t,1} (e_{i,1}e_{1,i}e_{t,1} - e_{1,1}^2e_{t,1}) \pmod{G_3}.$$

Clearly, for $t = 1$ the normal form of $\mathcal{R}_3^{(i,1,k,t)}$ coincides with $\mathcal{G}_{16}^{(i)}$. For $t \neq 1$, we combine the last result with the set $\{\mathcal{G}_6^{(i,t)} \mid \text{for all } 1 < i < t\} \subset G_3$ and obtain the set $\{\mathcal{G}_6^{(i,t)} \mid \text{for all } 1 \neq i \neq t \neq 1\}$. For $k < j$, we have

$$\begin{aligned} \mathcal{R}_4^{(i,j,k)} &= e_{i,j}e_{k,i}e_{i,k} - e_{i,k}e_{k,i}e_{i,j} + e_{i,j} \equiv e_{1,j}e_{k,1}e_{i,k} - e_{i,k}e_{k,1}e_{1,j} + e_{i,j} \pmod{G_1} \\ &\equiv \delta_{i,1}(e_{1,j}e_{k,1}e_{1,k} - e_{1,k}e_{k,1}e_{1,j}) + \widehat{\delta}_{i,1}(e_{1,j}e_{1,1}e_{i,1} - e_{i,1}e_{1,1}e_{1,j}) + e_{i,j} \pmod{G_1} \\ &\equiv \delta_{i,1}(e_{1,j}e_{k,1}e_{1,k} - e_{1,1}^2e_{1,j} + e_{1,j}) + \widehat{\delta}_{i,1}(e_{1,j}e_{1,1}e_{i,1} - e_{i,1}e_{1,1}e_{1,j} + e_{i,j}) \pmod{G_3}, \end{aligned}$$

using the relations $\mathcal{G}_2^{(i,j,k)}$, $\mathcal{G}_1^{(k,i,j)}$, $\mathcal{G}_2^{(k,1,i)}$, $\mathcal{G}_1^{(i,k,1)}$ and $\mathcal{G}_7^{(k,j)}$. For $i \neq 1$ the monic form of the last result coincides with $\mathcal{G}_0^{(j,i)}$. For $i = 1$, we combine the last result with the set $\{\mathcal{G}_8^{(j,k)} \mid \text{for all } 1 < j < k\} \subset G_3$ and obtain the set $\{\mathcal{G}_8^{(j,k)} \mid \text{for all } 1 \neq j \neq k\}$. For $r < i$, $j \neq k$ or $t \neq r$, and $s \neq k$ or $t \neq i$, we have

$$\begin{aligned} \mathcal{R}_5^{(i,j,k,t,r,s)} &= e_{i,j}e_{k,t}e_{r,s} - e_{r,s}e_{k,t}e_{i,j} \equiv \delta_{j,k}(e_{i,j}e_{j,t}e_{r,s} - e_{r,s}e_{j,t}e_{i,j}) \\ &+ \widehat{\delta}_{j,k}\delta_{i,t}(e_{i,j}e_{k,i}e_{r,s} - e_{r,s}e_{k,i}e_{i,j}) \pmod{G_1} \equiv \delta_{j,k}\left[\delta_{i,t}(e_{i,j}e_{j,i}e_{r,s} - e_{r,s}e_{j,i}e_{i,j})\right. \\ &+ \widehat{\delta}_{i,t}(e_{i,1}e_{1,t}e_{r,s} - e_{r,s}e_{1,t}e_{i,1})\left.\right] \pmod{G_1} \equiv \delta_{j,k}\left[\delta_{s,1}\widehat{\delta}_{i,t}(e_{i,1}e_{1,t}e_{r,1} - e_{r,1}e_{1,t}e_{i,1})\right] \\ &\pmod{G_1} \equiv 0 \pmod{G_3}, \end{aligned}$$

using the relations $\mathcal{G}_3^{(i,j,k,t)}$, $\mathcal{G}_3^{(k,t,i,j)}$, $\mathcal{G}_1^{(i,j,t)}$, $\mathcal{G}_2^{(j,t,i)}$, $\mathcal{G}_3^{(j,i,r,s)}$, $\mathcal{G}_3^{(r,s,j,i)}$, $\mathcal{G}_3^{(1,t,r,s)}$, $\mathcal{G}_3^{(r,s,1,t)}$, $\mathcal{G}_{11}^{(i,t,r)}$ and $\mathcal{G}_{11}^{(r,t,i)}$. Similarly, we can show that $\mathcal{R}_6^{(i,j,k,t,s)} \equiv 0 \pmod{G_1 \cup G_3}$. \square

The following lemma plays a crucial role in proving that the set \mathfrak{G} of Lemma 3.9 is a Gröbner basis for the ideal I .

Lemma 3.10. *For the universal enveloping algebra \mathfrak{A} , either*

$$(i) \dim \mathfrak{A} = \infty, \text{ or } (ii) \dim \mathfrak{A} < \infty \text{ and } \dim \mathfrak{A} \geq 4n^2 + 1.$$

Proof. Suppose that $\dim(\mathfrak{A}) < \infty$. We show that over an algebraically closed field F , there exist four inequivalent irreducible representations of degree n of the anti-Jordan triple system \mathfrak{J} , in addition to the trivial representation of degree 1. For $k = 1, \dots, 4$, we define the following maps:

$$\rho_k : \mathfrak{J} \rightarrow \text{End } V_k,$$

$$\rho_1(E_{i,j}) = E_{i,j}, \quad \rho_2(E_{i,j}) = -E_{i,j}, \quad \rho_3(E_{i,j}) = I E_{j,i}, \quad \rho_4(E_{i,j}) = -I E_{j,i},$$

where $I = \sqrt{-1}$. Our first step is to show that the maps ρ_k , $k = 1, \dots, 4$ are representations of the anti-Jordan triple system \mathfrak{J} . Clearly ρ_1 is a representation (the natural representation). For ρ_2 , we have

$$\rho_2(\langle E_{i,j}, E_{k,\ell}, E_{r,s} \rangle) = \rho_2(\delta_{j,k}\delta_{\ell,r}E_{i,s} - \delta_{s,k}\delta_{\ell,i}E_{r,j}) = -\delta_{j,k}\delta_{\ell,r}E_{i,s} + \delta_{s,k}\delta_{\ell,i}E_{r,j},$$

on the other hand, we have

$$\begin{aligned} \langle \rho_2(E_{i,j}), \rho_2(E_{k,\ell}), \rho_2(E_{r,s}) \rangle &= \rho_2(E_{i,j})\rho_2(E_{k,\ell})\rho_2(E_{r,s}) - \rho_2(E_{r,s})\rho_2(E_{k,\ell})\rho_2(E_{i,j}) \\ &= -\delta_{j,k}\delta_{\ell,r}E_{i,s} + \delta_{s,k}\delta_{\ell,i}E_{r,j}. \end{aligned}$$

Thus ρ_2 is a representation. For ρ_3 , we have

$$\rho_3(\langle E_{i,j}, E_{k,\ell}, E_{r,s} \rangle) = \rho_3(\delta_{j,k}\delta_{\ell,r}E_{i,s} - \delta_{s,k}\delta_{\ell,i}E_{r,j}) = \delta_{j,k}\delta_{\ell,r}I E_{s,i} - \delta_{s,k}\delta_{\ell,i}I E_{j,r},$$

on the other hand, we have

$$\begin{aligned} \langle \rho_3(E_{i,j}), \rho_3(E_{k,\ell}), \rho_3(E_{r,s}) \rangle &= \rho_3(E_{i,j})\rho_3(E_{k,\ell})\rho_3(E_{r,s}) - \rho_3(E_{r,s})\rho_3(E_{k,\ell})\rho_3(E_{i,j}) \\ &= -\delta_{i,\ell}\delta_{k,s}\mathbf{I}E_{j,r} + \delta_{r,\ell}\delta_{k,j}\mathbf{I}E_{s,i}. \end{aligned}$$

Similarly, we can show that ρ_4 is a representation. We now show that for all $i, j = 1, \dots, 4$ and $i \neq j$, the representations ρ_i and ρ_j are inequivalent. Indeed, there is no matrix T so that

$$\rho_i(x) = T^{-1}\rho_j(x)T, \quad \text{for all } x \in \mathfrak{J}, i \neq j.$$

This is easily seen by checking the trace on the both sides and using the definitions of the representations: $\text{Tr}(\rho_i(x)) \neq \text{Tr}(T^{-1}\rho_j(x)T) = \text{Tr}(\rho_j(x))$. The representations ρ_i , $i = 1, \dots, 4$ of \mathfrak{J} can be extended to representations of the universal envelope \mathfrak{A} . Hence \mathfrak{A} has a subalgebra of dimension $4n^2 + 1$, which is isomorphic to the direct sum of the matrix algebras corresponding to these representations. \square

We now can state the main theorem of this section.

Theorem 3.11. *With notation as above. If \mathfrak{J} is the anti-Jordan triple system of all $n \times n$ matrices ($n \geq 2$) then:*

- (i) *The set \mathfrak{G} is a Gröbner basis for the ideal I .*
- (ii) *The universal enveloping algebra \mathfrak{A} of \mathfrak{J} is finite-dimensional with basis \mathfrak{B} consists of $4n^2 + 1$ monomials:*

$$\begin{aligned} \mathfrak{B} = \{ &1, e_{i,j}, e_{i,1}e_{1,j}, e_{1,1}^2e_{1,j}, e_{1,1}^4 | i, j \in \Omega \} \cup \{ e_{1,i}e_{j,1} | i, j \in \Omega, (i, j) \neq (1, 1) \} \\ &\cup \{ e_{1,i}e_{1,1}e_{j,1} | i, j \in \Omega, j \neq 1 \}. \end{aligned}$$

Proof. By Lemma 3.9 the set \mathfrak{G} is the self-reduced form of the set $G \cup \bigcup_{i=1}^4 G_i$, so it remains to show that \mathfrak{G} is closed under any composition. We note first that there are $4n^2 + 1$ monomials of $F\langle X \rangle$ that do not have leading monomials of \mathfrak{G} as factors, namely,

$$1, \quad e_{i,j}, \quad e_{i,1}e_{1,j}, \quad e_{1,i}e_{j,1}, \quad (i, j) \neq (1, 1), \quad e_{1,i}e_{1,1}e_{j,1}, \quad j \neq 1, \quad e_{1,1}^2e_{1,j}, \quad e_{1,1}^4,$$

for all $i, j \in \Omega$. Suppose on the contrary that \mathfrak{G} is not a Gröbner basis for the ideal I . Then \mathfrak{G} is not closed under at least one composition by Theorem 2.6, i.e., there exist $f, g \in \mathfrak{G}$ such that $fx - yg \not\equiv 0 \pmod{\mathfrak{G}}$. We add the normal form of $fx - yg$ to the set \mathfrak{G} . Hence, the number of the monomials of $F\langle X \rangle$ that do not have the leading monomials of \mathfrak{G} as factors is less than $4n^2 + 1$. Hence, $\dim \mathfrak{A} < 4n^2 + 1$. But Lemma 3.10 implies that $\dim \mathfrak{A} \geq 4n^2 + 1$, which is a contradiction. This shows that \mathfrak{G} is a Gröbner basis for the ideal I . The proof (ii) is obvious by using (i) and Proposition 2.3. \square

4. THE STRUCTURE CONSTANTS OF \mathfrak{A}

In this section we use Theorem 3.11 and the relations of Lemma 3.9 to compute the structure constants of the universal enveloping algebra \mathfrak{A} .

Lemma 4.1. *Define an anti-automorphism $\eta: F\langle X \rangle \rightarrow F\langle X \rangle$ of the free associative algebra generated by $X = \{e_{i,j}\}_{i,j \in \Omega}$ by $\eta(e_{i,j}) = e_{j,i}$. Then η induces an anti-automorphism of order 2 on \mathfrak{A} (also denoted η).*

Proof. It suffices to show that the ideal $I = \langle \mathfrak{G} \rangle$ (see Theorem 3.11) is invariant under the action of η . We have, for example,

$$\begin{aligned}\eta\left(\mathcal{G}_0^{(i,j)}\right) &= e_{j,1}e_{1,1}e_{1,i} - e_{1,i}e_{1,1}e_{j,1} - e_{j,i} = \mathcal{G}_0^{(j,i)}, \\ \eta\left(\mathcal{G}_1^{(i,j,k)}\right) &= e_{k,j}e_{j,i} - e_{k,1}e_{1,i} = \mathcal{G}_1^{(k,j,i)}.\end{aligned}$$

A similar argument applies to all the other elements of \mathfrak{G} . \square

The next seven Propositions give the explicit structure constants of \mathfrak{A} .

Proposition 4.2. *Let $i, j, k, \ell \in \Omega$. Then in \mathfrak{A} , we have*

(5)

$$\begin{aligned}e_{i,j} \cdot e_{k,\ell} &= \delta_{j,k} \left[\delta_{i,\ell} \left\{ \left(\delta_{i,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{j,1} \right) e_{1,j} e_{j,1} + \left(\delta_{j,1} \widehat{\delta}_{i,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{j,1} \right) e_{i,1} e_{1,i} \right. \right. \\ &\quad \left. \left. - \widehat{\delta}_{i,1} \widehat{\delta}_{j,1} e_{1,1}^2 \right\} + \widehat{\delta}_{i,\ell} e_{i,1} e_{1,\ell} \right] + \widehat{\delta}_{j,k} \delta_{i,\ell} e_{1,j} e_{k,1},\end{aligned}$$

(6)

$$\begin{aligned}e_{i,j} \cdot e_{k,1} e_{1,\ell} &= \delta_{i,1} \left[\left(\delta_{j,k} \delta_{\ell,1} \left(\delta_{\ell,j} + \frac{1}{2} \widehat{\delta}_{\ell,j} \right) + \frac{1}{2} \widehat{\delta}_{j,k} \delta_{k,\ell} \delta_{j,1} \right) e_{1,1}^3 \right. \\ &\quad + \widehat{\delta}_{j,k} \delta_{k,\ell} \widehat{\delta}_{j,1} e_{1,1}^2 e_{1,j} + \delta_{j,k} \left(2 \delta_{\ell,j} \widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j} \left(\delta_{j,1} + \widehat{\delta}_{j,1} \widehat{\delta}_{\ell,1} \right) \right) e_{1,1}^2 e_{1,\ell} \\ &\quad + \frac{1}{2} \left(\delta_{\ell,1} \delta_{j,k} \widehat{\delta}_{\ell,j} - \widehat{\delta}_{j,k} \delta_{k,\ell} \delta_{j,1} \right) e_{1,1} - \left(\delta_{j,k} \delta_{\ell,j} \widehat{\delta}_{\ell,1} + \widehat{\delta}_{j,k} \delta_{k,\ell} \widehat{\delta}_{j,1} \right) e_{1,j} \Big] \\ &\quad + \widehat{\delta}_{i,1} \delta_{j,k} (e_{1,\ell} e_{1,1} e_{i,1} + e_{i,\ell}),\end{aligned}$$

(7)

$$\begin{aligned}e_{\ell,1} e_{1,k} \cdot e_{j,i} &= \delta_{i,1} \left[\left(\delta_{j,k} \delta_{\ell,1} \left(\delta_{\ell,j} + \frac{1}{2} \widehat{\delta}_{\ell,j} \right) + \frac{1}{2} \widehat{\delta}_{j,k} \delta_{k,\ell} \delta_{j,1} \right) e_{1,1}^3 + \widehat{\delta}_{j,k} \delta_{k,\ell} \widehat{\delta}_{j,1} e_{1,1}^2 e_{j,1} \right. \\ &\quad + \delta_{j,k} \left(2 \delta_{\ell,j} \widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j} \left(\delta_{j,1} + \widehat{\delta}_{j,1} \widehat{\delta}_{\ell,1} \right) \right) (e_{1,1}^2 e_{\ell,1} + e_{\ell,1}) \\ &\quad + \frac{1}{2} \left(\delta_{\ell,1} \delta_{j,k} \widehat{\delta}_{\ell,j} - \widehat{\delta}_{j,k} \delta_{k,\ell} \delta_{j,1} \right) e_{1,1} - \delta_{j,k} \delta_{\ell,j} \widehat{\delta}_{\ell,1} e_{j,1} \Big] \\ &\quad + \widehat{\delta}_{i,1} \delta_{j,k} \left(\delta_{\ell,1} e_{1,1}^2 e_{i,1} + \widehat{\delta}_{\ell,1} (e_{1,i} e_{1,1} e_{\ell,1} + e_{\ell,i}) \right),\end{aligned}$$

(8)

$$\begin{aligned}e_{i,j} \cdot e_{1,k} e_{\ell,1} &= \delta_{j,1} \left[\frac{1}{2} \left(\delta_{i,k} \delta_{\ell,1} \widehat{\delta}_{i,1} + \widehat{\delta}_{i,k} \widehat{\delta}_{\ell,1} \delta_{k,\ell} \delta_{i,1} \right) e_{1,1}^3 \right. \\ &\quad + \widehat{\delta}_{i,k} \delta_{k,\ell} \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} e_{1,1}^2 e_{i,1} + \delta_{i,k} \left(\delta_{i,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} \left(2 \delta_{i,\ell} + \widehat{\delta}_{i,\ell} \right) \right) e_{1,1}^2 e_{\ell,1} \\ &\quad + \frac{1}{2} \left(\widehat{\delta}_{i,k} \widehat{\delta}_{\ell,1} \delta_{k,\ell} \delta_{i,1} - \delta_{i,k} \delta_{\ell,1} \widehat{\delta}_{i,1} \right) e_{1,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} \left(\delta_{i,k} \delta_{i,\ell} + \widehat{\delta}_{i,k} \delta_{k,\ell} \right) e_{i,1} \Big] \\ &\quad + \widehat{\delta}_{j,1} \delta_{i,k} \left(\delta_{\ell,1} (e_{1,1}^2 e_{1,j} - e_{1,j}) + \widehat{\delta}_{\ell,1} e_{1,j} e_{1,1} e_{\ell,1} \right); \quad (k, \ell) \neq (1, 1),\end{aligned}$$

(9)

$$\begin{aligned}e_{1,\ell} e_{k,1} \cdot e_{j,i} &= \delta_{j,1} \left[\frac{1}{2} \left(\delta_{i,k} \delta_{\ell,1} \widehat{\delta}_{i,1} + \widehat{\delta}_{i,k} \widehat{\delta}_{\ell,1} \delta_{k,\ell} \delta_{i,1} \right) e_{1,1}^3 + \widehat{\delta}_{i,k} \delta_{k,\ell} \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} e_{1,1}^2 e_{1,i} \right. \\ &\quad + \delta_{i,k} \left(\delta_{i,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} \left(2 \delta_{i,\ell} + \widehat{\delta}_{i,\ell} \right) \right) (e_{1,1}^2 e_{1,\ell} - e_{1,\ell}) \\ &\quad + \frac{1}{2} \left(\widehat{\delta}_{i,k} \widehat{\delta}_{\ell,1} \delta_{k,\ell} \delta_{i,1} - \delta_{i,k} \delta_{\ell,1} \widehat{\delta}_{i,1} \right) e_{1,1} + \widehat{\delta}_{i,1} \widehat{\delta}_{\ell,1} \delta_{i,k} \delta_{i,\ell} e_{1,i} \Big] \\ &\quad + \widehat{\delta}_{j,1} \delta_{i,k} \left(\delta_{\ell,1} e_{1,1}^2 e_{j,1} + \widehat{\delta}_{\ell,1} e_{1,\ell} e_{1,1} e_{1,j} \right); \quad (k, \ell) \neq (1, 1).\end{aligned}$$

Proof. For (5), we use the relations $\mathcal{G}_3^{(i,j,k,\ell)}$, $\mathcal{G}_1^{(i,j,\ell)}$ and $\mathcal{G}_2^{(i,j,k)}$ and get

$$e_{i,j} \cdot e_{k,\ell} = \delta_{j,k} \left(\delta_{i,\ell} e_{i,j} e_{j,i} + \widehat{\delta}_{i,\ell} e_{i,1} e_{1,\ell} \right) + \widehat{\delta}_{j,k} \delta_{i,\ell} e_{1,j} e_{k,1}.$$

Using the relation $\mathcal{G}_4^{(i,j)}$ implies

$$\begin{aligned} e_{i,j} \cdot e_{k,\ell} &= \delta_{j,k} \left[\delta_{i,\ell} \left(\delta_{i,1} e_{1,j} e_{j,1} + \delta_{j,1} \widehat{\delta}_{i,1} e_{i,1} e_{1,i} + \widehat{\delta}_{i,1} \widehat{\delta}_{j,1} (e_{i,1} e_{1,i} + e_{1,j} e_{j,1} - e_{1,1}^2) \right) \right. \\ &\quad \left. + \widehat{\delta}_{i,\ell} e_{i,1} e_{1,\ell} \right] + \widehat{\delta}_{j,k} \delta_{i,\ell} e_{1,j} e_{k,1}. \end{aligned}$$

This completes the proof of (5). For (6), we use (5) (of the present proposition) and obtain

$$\begin{aligned} (e_{i,j} e_{k,1}) e_{1,\ell} &= \delta_{j,k} \left(\delta_{i,1} e_{1,j} e_{j,1} e_{1,\ell} + \widehat{\delta}_{i,1} e_{i,1} e_{1,1} e_{1,\ell} \right) + \widehat{\delta}_{j,k} \delta_{i,1} e_{1,j} e_{k,1} e_{1,\ell} \\ &= \delta_{i,1} \left(\delta_{j,k} e_{1,j} e_{j,1} e_{1,\ell} + \widehat{\delta}_{j,k} e_{1,j} e_{k,1} e_{1,\ell} \right) + \widehat{\delta}_{i,1} \delta_{j,k} e_{i,1} e_{1,1} e_{1,\ell}. \end{aligned}$$

We now write

$$A = e_{1,j} e_{j,1} e_{1,\ell}, \quad B = \widehat{\delta}_{j,k} e_{1,j} e_{k,1} e_{1,\ell},$$

and use the relations $\mathcal{G}_0^{(i,\ell)}$ (if $\ell \neq 1$) and $\mathcal{G}_5^{(i,1)}$ (if $\ell = 1$) for the last term to obtain

$$(10) \quad (e_{i,j} e_{k,1}) e_{1,\ell} = \delta_{i,1} (\delta_{j,k} A + B) + \widehat{\delta}_{i,1} \delta_{j,k} (e_{1,\ell} e_{1,1} e_{i,1} + e_{i,\ell}).$$

Using the relations $\mathcal{G}_{10}^{(j)}$, $\mathcal{G}_{15}^{(j)}$ and $\mathcal{G}_7^{(j,\ell)}$ gives

$$\begin{aligned} A &= \delta_{\ell,j} \left[\delta_{\ell,1} e_{1,1}^3 + \widehat{\delta}_{\ell,1} (2e_{1,1}^2 e_{1,j} - e_{1,j}) \right] \\ &\quad + \widehat{\delta}_{\ell,j} \left[\delta_{j,1} e_{1,1}^2 e_{1,\ell} + \widehat{\delta}_{j,1} \left(\delta_{\ell,1} \frac{1}{2} (e_{1,1}^3 + e_{1,1}) + \widehat{\delta}_{\ell,1} e_{1,1}^2 e_{1,\ell} \right) \right] \\ &= \delta_{\ell,1} \left[\left(\delta_{\ell,j} + \frac{1}{2} \widehat{\delta}_{\ell,j} \right) e_{1,1}^3 + \frac{1}{2} \widehat{\delta}_{\ell,j} e_{1,1} \right] + \left[2\delta_{\ell,j} \widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j} (\delta_{j,1} + \widehat{\delta}_{j,1} \widehat{\delta}_{\ell,1}) \right] e_{1,1}^2 e_{1,\ell} \\ &\quad - \delta_{\ell,j} \widehat{\delta}_{\ell,1} e_{1,j}. \end{aligned}$$

Using the relations $\mathcal{G}_{12}^{(j,k,\ell)}$, $\mathcal{G}_{14}^{(\ell)}$ and $\mathcal{G}_8^{(j,\ell)}$ gives

$$B = \widehat{\delta}_{j,k} \delta_{k,\ell} e_{1,j} e_{\ell,1} e_{1,\ell} = \widehat{\delta}_{j,k} \delta_{k,\ell} \left(\delta_{j,1} \frac{1}{2} (e_{1,1}^3 - e_{1,1}) + \widehat{\delta}_{j,1} (e_{1,1}^2 e_{1,j} - e_{1,j}) \right).$$

Using A and B in (10) and combining the coefficients completes the proof of (6). The proof of (7) is obvious by applying the anti-automorphism η (see Lemma 4.1) to both sides of (6) (of the present Proposition) and using the relations $\mathcal{G}_5^{(j,1)}$, $\mathcal{G}_5^{(\ell,1)}$ and $\mathcal{G}_8^{(i,1)}$. The proofs of (8) and (9) are similar. \square

Proposition 4.3. *Let $i, j, k, \ell \in \Omega$. Then in \mathfrak{A} , we have*

$$(11) \quad e_{i,j} \cdot e_{1,k} e_{1,1} e_{\ell,1} = \delta_{i,k} \left[-e_{1,j} e_{\ell,1} + \widehat{\delta}_{j,1} \delta_{j,\ell} \frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) \right]; \quad \ell \neq 1,$$

$$(12) \quad \begin{aligned} e_{i,j} \cdot e_{1,1}^2 e_{1,k} &= \delta_{j,1} \left[\delta_{i,1} \left(\delta_{k,1} e_{1,1}^4 + \widehat{\delta}_{k,1} e_{1,1} e_{1,k} \right) \right. \\ &\quad \left. + \widehat{\delta}_{i,1} \left(\delta_{i,k} \frac{1}{2} (e_{1,1}^4 - e_{1,1}^2) + e_{i,1} e_{1,k} \right) \right] - \widehat{\delta}_{j,1} \delta_{i,1} \delta_{k,1} e_{1,j} e_{1,1}, \end{aligned}$$

$$(13) \quad e_{i,j} \cdot e_{1,1}^4 = \delta_{j,1} \left[\delta_{i,1} e_{1,1} + \widehat{\delta}_{i,1} (e_{1,1}^2 e_{i,1} + e_{i,1}) \right] - \widehat{\delta}_{j,1} \delta_{i,1} (e_{1,1}^2 e_{1,j} - e_{1,j}).$$

Proof. For (11), let $\ell \neq 1$ and consider two cases. Case I. If $k = 1$ then (6) of Proposition 4.2 implies

$$(14) \quad e_{i,j}e_{1,1}^2 = \delta_{j,1} \left[\delta_{i,1}e_{1,1}^3 + \widehat{\delta}_{i,1}(e_{1,1}^2e_{i,1} + e_{i,1}) \right] + \widehat{\delta}_{j,1}\delta_{i,1} (e_{1,1}^2e_{1,j} - e_{1,j}).$$

Multiply (14) by $e_{\ell,1}$ and use the relations $\mathcal{G}_{18}^{(\ell)}$, $\mathcal{G}_3^{(i,1,\ell,1)}$, $\mathcal{G}_{11}^{(1,j,\ell)}$ and $\mathcal{G}_{13}^{(j)}$ and obtain

$$(e_{i,j}e_{1,1}^2) e_{\ell,1} = -\delta_{j,1}\delta_{i,1}e_{1,1}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{i,1} (\delta_{j,\ell}\frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) - e_{1,j}e_{\ell,1}).$$

Case II. If $k \neq 1$ then (8) of Proposition 4.2 implies

$$\begin{aligned} (e_{i,j}e_{1,k}e_{1,1}) e_{\ell,1} &= \delta_{j,1} \left[\delta_{i,k} \left(\delta_{i,1}e_{1,1}^3e_{\ell,1} + \widehat{\delta}_{i,1}\frac{1}{2} (e_{1,1}^3 - e_{1,1}) e_{\ell,1} \right) \right] \\ &\quad + \widehat{\delta}_{j,1}\delta_{i,k} (e_{1,1}^2e_{1,j} - e_{1,j}) e_{\ell,1}. \end{aligned}$$

Using the relations $\mathcal{G}_{18}^{(\ell)}$, $\mathcal{G}_{11}^{(1,j,\ell)}$ and $\mathcal{G}_{13}^{(j)}$ gives

$$\begin{aligned} (e_{i,j}e_{1,k}e_{1,1}) e_{\ell,1} &= \delta_{j,1} \left[\delta_{i,k} \left(-\delta_{i,1}e_{1,1}e_{\ell,1} + \widehat{\delta}_{i,1}\frac{1}{2} (-e_{1,1} - e_{1,1})e_{\ell,1} \right) \right] \\ &\quad + \widehat{\delta}_{j,1}\delta_{i,k} [\delta_{j,\ell}\frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) - e_{1,j}e_{\ell,1}] \\ &= -\delta_{j,1}\delta_{i,k}e_{1,1}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{i,k} [\delta_{j,\ell}\frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) - e_{1,j}e_{\ell,1}]. \end{aligned}$$

Combining the results of the two cases completes the proof of (11). For (12), we multiply (14) by $e_{1,k}$ and use the relations $\mathcal{G}_{17}^{(k)}$, $\mathcal{G}_{12}^{(1,i,k)}$, $\mathcal{G}_{14}^{(i)}$, $\mathcal{G}_{11}^{(1,j,1)}$ and $\mathcal{G}_3^{(1,j,1,k)}$. The proof of (13) is similar. \square

The proofs of the next five Propositions are similar to the proofs of Propositions 4.2 and 4.3 and are omitted.

Proposition 4.4. *Let $i, j, k, \ell \in \Omega$. Then in \mathfrak{A} , we have*

$$\begin{aligned} (15) \quad e_{i,1}e_{1,j} \cdot e_{k,1}e_{1,\ell} &= \delta_{j,k}\delta_{\ell,1} \left(\delta_{\ell,j}\delta_{i,1}e_{1,1}^4 + \frac{1}{2}\widehat{\delta}_{\ell,j}\delta_{i,1} (e_{1,1}^4 + e_{1,1}^2) \right) \\ &\quad + \frac{1}{2} \left[\delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell} \left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j} (\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}) \right) \right. \\ &\quad \left. + \widehat{\delta}_{j,k}\delta_{k,\ell} \left(\widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j} + \delta_{j,1}\delta_{i,1} \right) \right] (e_{1,1}^4 - e_{1,1}^2) \\ &\quad + \delta_{j,k} \left(\delta_{\ell,1}\widehat{\delta}_{i,1} + \delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j} (\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}) \right) e_{i,1}e_{1,\ell}, \end{aligned}$$

$$\begin{aligned} (16) \quad e_{i,1}e_{1,j} \cdot e_{1,k}e_{\ell,1} &= \delta_{j,1}\widehat{\delta}_{k,1}\delta_{k,\ell}\frac{1}{2} \left(\delta_{i,1} (e_{1,1}^4 + e_{1,1}^2) + 2\widehat{\delta}_{i,1}e_{i,1}e_{1,1} \right) \\ &\quad - \delta_{k,1} \left(\delta_{j,1}\delta_{i,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\delta_{i,j} \right) e_{1,1}e_{\ell,1}; \quad (k, \ell) \neq (1, 1), \end{aligned}$$

$$(17) \quad e_{i,1}e_{1,j} \cdot e_{1,k}e_{1,1}e_{\ell,1} = -\delta_{k,1} \left(\delta_{i,1}\delta_{j,1} + \widehat{\delta}_{j,1}\delta_{i,j} \right) e_{1,1}^2e_{\ell,1}; \quad \ell \neq 1,$$

$$\begin{aligned} (18) \quad e_{i,1}e_{1,j} \cdot e_{1,1}^2e_{1,k} &= \delta_{j,1} \left[\delta_{k,1} \left(\delta_{i,1}e_{1,1} + \widehat{\delta}_{i,1} (e_{1,1}^2e_{i,1} + e_{i,1}) \right) \right. \\ &\quad \left. + \widehat{\delta}_{k,1} \left(\delta_{i,1}e_{1,1}^2e_{1,k} + \widehat{\delta}_{i,1} (e_{1,k}e_{1,1}e_{i,1} + e_{i,k}) \right) \right] \\ &\quad - \widehat{\delta}_{j,1}\delta_{k,1}\delta_{i,j}\frac{1}{2} (e_{1,1}^3 - e_{1,1}), \end{aligned}$$

$$(19) \quad e_{i,1}e_{1,j} \cdot e_{1,1}^4 = \delta_{j,1}e_{i,1}e_{1,1} + \widehat{\delta}_{j,1}\delta_{j,i}\frac{1}{2} (e_{1,1}^2 - e_{1,1}^4).$$

Proposition 4.5. *Let $i, j, k, \ell \in \Omega$ and $(i, j) \neq (1, 1)$. Then in \mathfrak{A} , we have*

(20)

$$\begin{aligned} e_{1,i}e_{j,1} \cdot e_{1,k}e_{\ell,1} &= \frac{1}{2} \left[\delta_{i,1}\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1} (e_{1,1}^4 - e_{1,1}^2) \right. \\ &\quad + \left\{ \delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell} \left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1} \left(2\delta_{j,\ell} + \widehat{\delta}_{j,\ell} \right) \right) \right. \\ &\quad \left. + \widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell} \left(\widehat{\delta}_{j,1}\widehat{\delta}_{j,\ell}\delta_{i,j} + \delta_{i,1}\delta_{j,1} \right) \right\} (e_{1,1}^4 + e_{1,1}^2) \left. \right] \\ &\quad - \delta_{j,k} \left(\delta_{j,1} + \widehat{\delta}_{j,1} \left(\delta_{\ell,1}\widehat{\delta}_{i,1} + \widehat{\delta}_{\ell,1} \right) \right) e_{1,i}e_{\ell,1}; \quad (k, \ell) \neq (1, 1), \end{aligned}$$

(21)

$$\begin{aligned} e_{1,i}e_{j,1} \cdot e_{1,1}^2e_{1,k} &= \left(-\delta_{j,1}\widehat{\delta}_{i,1}\delta_{k,1} + \widehat{\delta}_{j,1}\delta_{i,j}\delta_{k,i} \right) e_{1,1}^2e_{1,i} + \delta_{j,1}\delta_{k,1}\widehat{\delta}_{i,1}e_{1,i} \\ &\quad + \delta_{i,j}\widehat{\delta}_{j,1} \left(\frac{1}{2}\delta_{k,1} (e_{1,1}^3 + e_{1,1}) + \widehat{\delta}_{k,i}\widehat{\delta}_{k,1}e_{1,1}^2e_{1,k} \right), \end{aligned}$$

$$(22) \quad e_{1,i}e_{j,1} \cdot e_{1,1}^4 = \delta_{j,1}e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j}\frac{1}{2}(e_{1,1}^4 + e_{1,1}^2),$$

(23)

$$e_{1,i}e_{j,1} \cdot e_{1,k}e_{1,1}e_{\ell,1} = -\delta_{j,k}e_{1,i}e_{1,1}e_{\ell,1}; \quad \ell \neq 1.$$

Proposition 4.6. *Let $i, j, k \in \Omega$. Then in \mathfrak{A} , we have*

$$(24) \quad \begin{aligned} e_{1,1}^2e_{1,k} \cdot e_{i,j} &= \delta_{k,i} \left[\delta_{j,1} \left(\left(\delta_{k,1} + \frac{1}{2}\widehat{\delta}_{k,1} \right) e_{1,1}^4 + \frac{1}{2}\widehat{\delta}_{k,1}e_{1,1}^2 \right) + \widehat{\delta}_{j,1}e_{1,1}e_{1,j} \right] \\ &\quad - \widehat{\delta}_{k,i}\delta_{j,1}\delta_{k,1}e_{1,1}e_{i,1}, \end{aligned}$$

$$(25) \quad \begin{aligned} e_{1,1}^2e_{1,k} \cdot e_{i,1}e_{1,j} &= \left[\delta_{k,i}\delta_{j,1} \left(\delta_{k,1} + \frac{1}{2}\widehat{\delta}_{k,1} \right) + \frac{1}{2}\widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j} \right] e_{1,1} \\ &\quad + \frac{1}{2} \left(\delta_{k,i}\widehat{\delta}_{k,1}\delta_{j,1} - \widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j} \right) e_{1,1}^3 \\ &\quad + \left(\delta_{k,1} \left(\delta_{k,i}\widehat{\delta}_{j,1} - \widehat{\delta}_{k,i}\delta_{i,1} \right) + \widehat{\delta}_{k,1}\widehat{\delta}_{j,1} \right) e_{1,1}^2e_{1,j}, \end{aligned}$$

$$(26) \quad e_{1,1}^2e_{1,k} \cdot e_{i,1}e_{j,1} = \delta_{k,1} \left(-\delta_{i,1}e_{1,1}^2e_{j,1} + \widehat{\delta}_{i,1}\delta_{i,j}\frac{1}{2}(e_{1,1}^3 + e_{1,1}) \right); \quad (i, j) \neq (1, 1),$$

$$(27) \quad e_{1,1}^2e_{1,k} \cdot e_{1,j}e_{1,1}e_{\ell,1} = \delta_{k,1}\delta_{j,1}e_{1,1}e_{\ell,1}; \quad \ell \neq 1,$$

$$(28) \quad e_{1,1}^2e_{1,k} \cdot e_{1,1}^2e_{1,j} = \delta_{k,1}e_{1,1}e_{1,j},$$

$$(29) \quad e_{1,1}^2e_{1,j} \cdot e_{1,1}^4 = \delta_{j,1}e_{1,1}^3.$$

Proposition 4.7. *Let $i, j, k, \ell \in \Omega$ and $\ell \neq 1$. Then in \mathfrak{A} , we have*

$$(30) \quad \begin{aligned} e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,j} &= \delta_{j,\ell} \left[\frac{1}{2} \left\{ \left(\delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1} + \widehat{\delta}_{i,1}\delta_{i,k} \right) e_{1,1}^4 \right. \right. \\ &\quad \left. \left. + \left(\widehat{\delta}_{i,1}\delta_{i,k} - \delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1} \right) e_{1,1}^2 \right\} - \widehat{\delta}_{i,1}\delta_{k,1}e_{1,1}e_{i,1} \right. \\ &\quad \left. - \widehat{\delta}_{k,1}e_{1,k}e_{i,1} \right], \end{aligned}$$

$$(31) \quad \begin{aligned} e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,i}e_{j,1} &= \delta_{i,\ell} \left[\delta_{k,1} \left(\delta_{j,1}\frac{1}{2}(e_{1,1} - e_{1,1}^3) - \widehat{\delta}_{j,1}e_{1,1}^2e_{j,1} \right) \right. \\ &\quad \left. - \widehat{\delta}_{k,1} \left(\widehat{\delta}_{j,1}e_{1,k}e_{1,1}e_{j,1} + \delta_{j,1}(e_{1,1}^2e_{1,k} - e_{1,k}) \right) \right]; \\ &\quad (i, j) \neq (1, 1), \end{aligned}$$

$$(32) \quad e_{1,i}e_{1,1}e_{j,1} \cdot e_{1,k}e_{1,1}e_{\ell,1} = \delta_{j,k} \left(e_{1,i}e_{\ell,1} - \widehat{\delta}_{i,1}\delta_{i,\ell}\frac{1}{2}(e_{1,1}^4 + e_{1,1}^2) \right); \quad j \neq 1,$$

$$(33) \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,1}e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^2e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^4 = 0.$$

Proposition 4.8. *Let $i, j \in \Omega$. Then in \mathfrak{A} , we have*

$$(34) \quad e_{1,1}^4 \cdot e_{i,j} = \delta_{i,1} \left(\delta_{j,1}e_{1,1} + \widehat{\delta}_{j,1}e_{1,1}^2e_{1,j} \right) - \widehat{\delta}_{i,1}\delta_{j,1}e_{1,1}^2e_{j,1},$$

$$(35) \quad e_{1,1}^4 \cdot e_{1,i}e_{j,1} = \delta_{i,1}e_{1,1}e_{j,1} + \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\delta_{i,j}\frac{1}{2}(e_{1,1}^4 + e_{1,1}^2); \quad (i, j) \neq (1, 1),$$

$$(36) \quad e_{1,1}^4 \cdot e_{i,1}e_{1,j} = \delta_{i,1}e_{1,1}e_{1,j} + \widehat{\delta}_{i,1}\delta_{i,j}\frac{1}{2}(e_{1,1}^2 - e_{1,1}^4),$$

$$(37) \quad e_{1,1}^4 \cdot e_{1,1}^2e_{1,j} = e_{1,1}^2e_{1,j},$$

$$(38) \quad e_{1,1}^4 \cdot e_{1,i}e_{1,1}e_{j,1} = \delta_{i,1}e_{1,1}^2e_{j,1}; \quad j \neq 1,$$

$$(39) \quad e_{1,1}^4 \cdot e_{1,1}^4 = e_{1,1}^4.$$

5. THE CENTER OF THE UNIVERSAL ENVELOPING ALGEBRA \mathfrak{A}

Our next aim is to use the results of Section 4 to determine the center of \mathfrak{A} :

$$Z(\mathfrak{A}) = \{z \in \mathfrak{A} \mid zu = uz, \text{ for all } u \in \mathfrak{A}\}.$$

Theorem 5.1. *The center $Z(\mathfrak{A})$ of the (unital) universal enveloping algebra \mathfrak{A} has dimension 5 with basis:*

$$\begin{aligned} z_1 &= \frac{(n-2)}{n}e_{1,1}^2 - \frac{2}{n} \sum_{i=2}^n e_{1,i}e_{i,1} + e_{1,1}^4, & z_2 &= (2-n)e_{1,1}^2 + \sum_{i=2}^n e_{1,i}e_{i,1} + \sum_{i=2}^n e_{i,1}e_{1,i}, \\ z_3 &= -\frac{1}{2}e_{1,1} + \frac{1}{2}e_{1,1}^3 + \sum_{i=2}^n e_{1,i}e_{1,1}e_{i,1}, & z_4 &= \sum_{i=1}^n e_{i,i}, & z_5 &= 1. \end{aligned}$$

Proof. To get the center of \mathfrak{A} , it is sufficient to determine the elements of \mathfrak{A} which commute with $e_{i,j}$, for all $i, j \in \Omega$. Let

$$\begin{aligned} x &= \sum_{i,j=1}^n \zeta_1^{(i,j)} e_{i,j} + \sum_{i=1}^n \sum_{j=2}^n \zeta_2^{(i,j)} e_{1,i}e_{1,1}e_{j,1} + \sum_{j=1}^n \zeta_3^{(j)} e_{1,1}^2e_{1,j} \\ &\quad + \sum_{i,j=1}^n \zeta_4^{(i,j)} e_{i,1}e_{1,j} + \sum_{\substack{i,j=1 \\ (i,j) \neq (1,1)}}^n \zeta_5^{(i,j)} e_{1,i}e_{j,1} + \zeta e_{1,1}^4, \end{aligned}$$

be any element of $Z(\mathfrak{A})$. Then

$$\begin{aligned} (40) \quad 0 &= x e_{1,1} - e_{1,1} x \\ &= \sum_{i,j=1}^n \zeta_1^{(i,j)} (e_{i,j}e_{1,1} - e_{1,1}e_{i,j}) + \sum_{i=1}^n \sum_{j=2}^n \zeta_2^{(i,j)} (e_{1,i}e_{1,1}e_{j,1}e_{1,1} - e_{1,1}e_{1,i}e_{1,1}e_{j,1}) \\ &\quad + \sum_{j=1}^n \zeta_3^{(j)} (e_{1,1}^2e_{1,j}e_{1,1} - e_{1,1}^3e_{1,j}) + \sum_{i,j=1}^n \zeta_4^{(i,j)} (e_{i,1}e_{1,j}e_{1,1} - e_{1,1}e_{i,1}e_{1,j}) \\ &\quad + \sum_{\substack{i,j=1 \\ (i,j) \neq (1,1)}}^n \zeta_5^{(i,j)} (e_{1,i}e_{j,1}e_{1,1} - e_{1,1}e_{1,i}e_{j,1}). \end{aligned}$$

Proposition 4.2 implies that $e_{i,1}e_{1,j}e_{1,1} = 0 = e_{1,i}e_{j,1}e_{1,1}$ for $i \neq j \neq 1$, $e_{1,1}e_{i,1}e_{1,j} = 0 = e_{1,1}e_{1,i}e_{j,1}$ for $1 \neq i \neq j$, $e_{i,1}e_{1,i}e_{1,1} = \frac{1}{2}(e_{1,1}^3 - e_{1,1}) = e_{1,1}e_{i,1}e_{1,i}$ and $e_{1,i}e_{i,1}e_{1,1} = \frac{1}{2}(e_{1,1}^3 + e_{1,1}) = e_{1,1}e_{1,i}e_{i,1}$ for $i \neq 1$. Using this in (40) gives

$$\begin{aligned} 0 &= \sum_{i,j=1}^n \zeta_1^{(i,j)} (e_{i,j}e_{1,1} - e_{1,1}e_{i,j}) - \sum_{j=2}^n \zeta_2^{(1,j)} e_{1,1}^3 e_{j,1} - \sum_{j=2}^n \zeta_3^{(j)} e_{1,1}^3 e_{1,j} \\ &\quad + \sum_{i=2}^n \zeta_4^{(i,1)} e_{i,1}e_{1,1}^2 - \sum_{j=2}^n \zeta_4^{(1,j)} e_{1,1}^2 e_{1,j} + \sum_{i=2}^n \zeta_5^{(i,1)} e_{1,i}e_{1,1}^2 - \sum_{j=2}^n \zeta_5^{(1,j)} e_{1,1}^2 e_{j,1}. \end{aligned}$$

Using (5), (6) of Proposition 4.2 and (11), (12) of Proposition 4.3 implies

$$\begin{aligned} 0 &= \sum_{j=2}^n \zeta_1^{(1,j)} (e_{1,j}e_{1,1} - e_{1,1}e_{1,j}) + \sum_{i=2}^n \zeta_1^{(i,1)} (e_{i,1}e_{1,1} - e_{1,1}e_{i,1}) \\ &\quad + \sum_{j=2}^n \zeta_2^{(1,j)} e_{1,1}e_{j,1} - \sum_{j=2}^n \zeta_3^{(j)} e_{1,1}e_{1,j} + \sum_{i=2}^n \zeta_4^{(i,1)} (e_{1,1}^2 e_{i,1} + e_{i,1}) \\ &\quad - \sum_{j=2}^n \zeta_4^{(1,j)} e_{1,1}^2 e_{1,j} + \sum_{i=2}^n \zeta_5^{(i,1)} (e_{1,1}^2 e_{1,i} - e_{1,i}) - \sum_{j=2}^n \zeta_5^{(1,j)} e_{1,1}^2 e_{j,1}. \end{aligned}$$

Comparing the coefficients on both sides, we get

$$\zeta_1^{(1,j)} = \zeta_1^{(i,1)} = \zeta_2^{(1,j)} = \zeta_3^{(j)} = \zeta_4^{(i,1)} = \zeta_4^{(1,i)} = \zeta_5^{(i,1)} = \zeta_5^{(1,j)} = 0,$$

for all $i, j \in \Omega \setminus \{1\}$. Rewriting x with these values for the coefficients, we obtain

$$\begin{aligned} x &= \zeta_1^{(1,1)} e_{1,1} + \sum_{i,j=2}^n \zeta_1^{(i,j)} e_{i,j} + \sum_{i,j=2}^n \zeta_2^{(i,j)} e_{1,i}e_{1,1}e_{j,1} + \zeta_3^{(1)} e_{1,1}^3 \\ &\quad + \zeta_4^{(1,1)} e_{1,1}^2 + \sum_{i,j=2}^n \zeta_4^{(i,j)} e_{i,1}e_{1,j} + \sum_{i,j=2}^n \zeta_5^{(i,j)} e_{1,i}e_{j,1} + \zeta_{1,1}^4. \end{aligned}$$

Choose $q \neq 1$ and observe that $e_{1,1}e_{q,q} = 0 = e_{q,q}e_{1,1}$ by (5) of Proposition 4.2. Hence,

$$\begin{aligned} 0 &= x e_{q,q} - e_{q,q} x \\ &= \sum_{i,j=2}^n \zeta_1^{(i,j)} (e_{i,j}e_{q,q} - e_{q,q}e_{i,j}) + \sum_{i,j=2}^n \zeta_2^{(i,j)} (e_{1,i}e_{1,1}e_{j,1}e_{q,q} - e_{q,q}e_{1,i}e_{1,1}e_{j,1}) \\ &\quad + \sum_{i,j=2}^n \zeta_4^{(i,j)} (e_{i,1}e_{1,j}e_{q,q} - e_{q,q}e_{i,1}e_{1,j}) + \sum_{i,j=2}^n \zeta_5^{(i,j)} (e_{1,i}e_{j,1}e_{q,q} - e_{q,q}e_{1,i}e_{j,1}). \end{aligned}$$

Using Proposition 4.2, (30) of Proposition 4.7 and (11) of Proposition 4.3 implies

$$\begin{aligned}
0 = & \sum_{\substack{i=2 \\ i \neq q}}^n \zeta_1^{(i,q)} (e_{i,1}e_{1,q} - e_{1,q}e_{i,1}) + \sum_{\substack{j=2 \\ j \neq q}}^n \zeta_1^{(q,j)} (e_{1,j}e_{q,1} - e_{q,1}e_{1,j}) \\
& - \sum_{\substack{i=2 \\ i \neq q}}^n \zeta_2^{(i,q)} e_{1,i}e_{q,1} + \sum_{\substack{j=2 \\ j \neq q}}^n \zeta_2^{(q,j)} e_{1,q}e_{j,1} + \sum_{\substack{i=2 \\ i \neq q}}^n \zeta_4^{(i,q)} (e_{1,q}e_{1,1}e_{i,1} + e_{i,q}) \\
& - \sum_{\substack{j=2 \\ j \neq q}}^n \zeta_4^{(q,j)} (e_{1,j}e_{1,1}e_{q,1} + e_{q,j}) + \sum_{\substack{i=2 \\ i \neq q}}^n \zeta_5^{(i,q)} e_{1,i}e_{1,1}e_{q,1} - \sum_{\substack{j=2 \\ j \neq q}}^n \zeta_5^{(q,j)} e_{1,q}e_{1,1}e_{j,1}.
\end{aligned}$$

Comparing the coefficients on both sides gives

$$\zeta_1^{(i,q)} = \zeta_1^{(q,j)} = \zeta_2^{(i,q)} = \zeta_2^{(q,j)} = \zeta_4^{(i,q)} = \zeta_4^{(q,j)} = \zeta_5^{(i,q)} = \zeta_5^{(q,j)} = 0,$$

for all $i, j, q \in \Omega \setminus \{1\}$ and $i \neq q \neq j$. Rewriting x with these values for the coefficients, we get

$$\begin{aligned}
x = & \sum_{i=1}^n \left(\zeta_1^{(i,i)} e_{i,i} + \zeta_4^{(i,i)} e_{i,1}e_{1,i} \right) \\
& + \sum_{i=2}^n \left(\zeta_2^{(i,i)} e_{1,i}e_{1,1}e_{i,1} + \zeta_5^{(i,i)} e_{1,i}e_{i,1} \right) + \zeta_3^{(1)} e_{1,1}^3 + \zeta e_{1,1}^4.
\end{aligned}$$

We next choose $q, s \in \Omega \setminus \{1\}$ and $q \neq s$ and observe that $e_{1,1}e_{q,s} = 0 = e_{q,s}e_{1,1}$ by (5) of Proposition 4.2. Hence,

$$\begin{aligned}
0 = & xe_{qs} - e_{q,s}x \\
= & \sum_{i=1}^n \left(\zeta_1^{(i,i)} (e_{i,i}e_{q,s} - e_{q,s}e_{i,i}) + \zeta_4^{(i,i)} (e_{i,1}e_{1,i}e_{q,s} - e_{q,s}e_{i,1}e_{1,i}) \right) \\
& + \sum_{i=2}^n \left(\zeta_2^{(i,i)} (e_{1,i}e_{1,1}e_{i,1}e_{q,s} - e_{q,s}e_{1,i}e_{1,1}e_{i,1}) + \zeta_5^{(i,i)} (e_{1,i}e_{i,1}e_{q,s} - e_{q,s}e_{1,i}e_{i,1}) \right).
\end{aligned}$$

Using Proposition 4.2, (30) of Proposition 4.7 and (11) of Proposition 4.3 gives

$$\begin{aligned}
0 = & \left(\zeta_1^{(q,q)} - \zeta_1^{(s,s)} \right) e_{q,1}e_{1,s} + \left(\zeta_1^{(s,s)} - \zeta_1^{(q,q)} \right) e_{1,s}e_{q,1} \\
& + \left(\zeta_4^{(q,q)} - \zeta_4^{(s,s)} \right) (e_{1,s}e_{1,1}e_{q,1} + e_{q,s}) \\
& - \left(\zeta_2^{(s,s)} - \zeta_2^{(q,q)} \right) e_{1,s}e_{q,1} + \left(\zeta_5^{(s,s)} - \zeta_5^{(q,q)} \right) e_{1,s}e_{1,1}e_{q,1}.
\end{aligned}$$

Comparing the coefficients on both sides gives

$$\zeta_1^{(q,q)} = \zeta_1^{(s,s)}, \quad \zeta_4^{(q,q)} = \zeta_4^{(s,s)}, \quad \zeta_2^{(s,s)} = \zeta_2^{(q,q)}, \quad \zeta_5^{(s,s)} = \zeta_5^{(q,q)},$$

for all $q, s \in \Omega \setminus \{1\}$ and $q \neq s$. Hence the values of $\zeta_1^{(q,q)}, \zeta_4^{(q,q)}, \zeta_2^{(q,q)}$ and $\zeta_5^{(q,q)}$ (for all $q \in \Omega \setminus \{1\}$) do not depend on the value of q . We remove the exponents of

$\zeta_1^{(q,q)}, \zeta_4^{(q,q)}, \zeta_2^{(q,q)}$ and $\zeta_5^{(q,q)}$ and rewrite x and obtain

$$\begin{aligned} x &= \zeta_1^{(1,1)} e_{1,1} + \zeta_1 \sum_{i=2}^n e_{i,i} + \zeta_4^{(1,1)} e_{1,1}^2 \\ &\quad + \sum_{i=2}^n (\zeta_4 e_{i,1} e_{1,i} + \zeta_2 e_{1,i} e_{1,1} e_{i,1} + \zeta_5 e_{1,i} e_{i,1}) + \zeta_3^{(1)} e_{1,1}^3 + \zeta e_{1,1}^4. \end{aligned}$$

We observe that x is invariant under the action of the anti-automorphism η . Therefore, if x commutes with $e_{1,q}$ (resp. $e_{q,1}$) ($q \neq 1$) then x commutes with $e_{q,1}$ (resp. $e_{1,q}$). Choose $q \neq \Omega \setminus \{1\}$ and note that $e_{q,1} e_{i,1} = 0 = e_{i,1} e_{q,1}$ ($i \neq 1$) by (5) of proposition 4.2. Hence,

$$\begin{aligned} 0 &= x e_{q,1} - e_{q,1} x \\ &= \zeta_1^{(1,1)} (e_{1,1} e_{q,1} - e_{q,1} e_{1,1}) + \zeta_1 \sum_{i=2}^n (e_{i,i} e_{q,1} - e_{q,1} e_{i,i}) \\ &\quad + \zeta_4^{(1,1)} (e_{1,1}^2 e_{q,1} - e_{q,1} e_{1,1}^2) + \zeta_4 \sum_{i=2}^n e_{i,1} e_{1,i} e_{q,1} - \zeta_2 \sum_{i=2}^n e_{q,1} e_{1,i} e_{1,1} e_{i,1} \\ &\quad - \zeta_5 \sum_{i=2}^n e_{q,1} e_{1,i} e_{i,1} + \zeta_3^{(1)} (e_{1,1}^3 e_{q,1} - e_{q,1} e_{1,1}^3) + \zeta (e_{1,1}^4 e_{q,1} - e_{q,1} e_{1,1}^4). \end{aligned}$$

Using (5)-(7) of Proposition 4.2, (24) of Propositions 4.6, (34) of Proposition 4.8 and Proposition 4.3 gives

$$\begin{aligned} 0 &= \zeta_1^{(1,1)} (e_{1,1} e_{q,1} - e_{q,1} e_{1,1}) + \zeta_1 \sum_{i=2}^n (e_{q,1} e_{1,i} - e_{1,i} e_{q,1}) - \zeta_4^{(1,1)} e_{q,1} \\ &\quad + \zeta_4 (n e_{1,1}^2 e_{q,1} + e_{q,1}) + \zeta_2 e_{1,1} e_{q,1} - \zeta_5 (n e_{1,1}^2 e_{q,1} + (n-1) e_{q,1}) \\ &\quad + \zeta_3^{(1)} (-e_{1,1} e_{q,1} - e_{q,1} e_{1,1}) - \zeta (2 e_{1,1}^2 e_{q,1} + e_{q,1}). \end{aligned}$$

Combining the coefficients gives

$$\begin{aligned} 0 &= \left(\zeta_1^{(1,1)} - \zeta_1 + \zeta_2 - \zeta_3^{(1)} \right) e_{1,1} e_{q,1} + \left(-\zeta_1^{(1,1)} + \zeta_1 - \zeta_3^{(1)} \right) e_{q,1} e_{1,1} \\ &\quad + \left(-\zeta_4^{(1,1)} + \zeta_4 - (n-1) \zeta_5 - \zeta \right) e_{q,1} + (n \zeta_4 - n \zeta_5 - 2 \zeta) e_{1,1}^2 e_{q,1}. \end{aligned}$$

Comparing the coefficients on both sides gives

$$\begin{aligned} \zeta_1^{(1,1)} - \zeta_1 + \zeta_2 - \zeta_3^{(1)} &= 0, & -\zeta_1^{(1,1)} + \zeta_1 - \zeta_3^{(1)} &= 0, \\ -\zeta_4^{(1,1)} + \zeta_4 - (n-1) \zeta_5 - \zeta &= 0, & n \zeta_4 - n \zeta_5 - 2 \zeta &= 0. \end{aligned}$$

These equations can be reduced to the system

$$\begin{aligned} \zeta_1^{(1,1)} - \zeta_1 + \zeta_3^{(1)} &= 0, & \zeta_2 - 2 \zeta_3^{(1)} &= 0, \\ \zeta_4^{(1,1)} + (n-2) \zeta_4 - \left(\frac{n-2}{n} \right) \zeta &= 0, & \zeta_5 - \zeta_4 + \frac{2}{n} \zeta &= 0. \end{aligned}$$

This is a linear system of four equations in eight variables. Hence, there are four free variables. Setting,

$$(\zeta, \zeta_4, \zeta_2, \zeta_1) = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),$$

in the last system gives

$$\left(\zeta_1^{(1,1)}, \zeta_3^{(1)}, \zeta_4^{(1,1)}, \zeta_5 \right) = \left(0, 0, \frac{n-2}{n}, \frac{-2}{n} \right), (0, 0, 2-n, 1), \left(-\frac{1}{2}, \frac{1}{2}, 0, 0 \right), (1, 0, 0, 0),$$

respectively. Using these solutions in x gives z_1, z_2, z_3, z_4 respectively. \square

6. EXPLICIT DECOMPOSITION OF THE UNIVERSAL ENVELOPING ALGEBRA

Theorem 6.1. *The universal enveloping algebra \mathfrak{A} of the anti-Jordan triple system \mathfrak{J} can be decomposed as follows:*

$$\mathfrak{A} = F \oplus M_{n,n}(F) \oplus M_{n,n}(F) \oplus M_{n,n}(F) \oplus M_{n,n}(F),$$

where $M_{n,n}$ is the ordinary associative algebra of all $n \times n$ matrices.

Proof. We define the first two sets of $n \times n$ matrix units. For all $k \in \{0, 1\}$ and $i, j = 2, \dots, n$, we set

$$\begin{aligned} B_{1,1}^{(k)} &= \frac{1}{4} (e_{1,1}^4 + e_{1,1}^2 + (-1)^k (e_{1,1}^3 + e_{1,1})), \\ B_{1,i}^{(k)} &= e_{1,1}e_{1,i} + (-1)^k e_{1,1}^2 e_{1,i}, \\ B_{i,1}^{(k)} &= \frac{1}{4} (e_{i,1}e_{1,1} + (-1)^k (e_{1,1}^2 e_{i,1} + e_{i,1})), \\ B_{i,i}^{(k)} &= \frac{1}{2} \left(\frac{1}{2} (e_{1,1}^4 - e_{1,1}^2) + e_{i,1}e_{1,i} + (-1)^k (e_{1,i}e_{1,1}e_{i,1} + e_{i,i}) \right), \\ B_{i,j}^{(k)} &= \frac{1}{2} (e_{i,1}e_{1,j} + (-1)^k (e_{1,j}e_{1,1}e_{i,1} + e_{i,j})); \quad i \neq j. \end{aligned}$$

We wish to show that for each $k \in \{0, 1\}$, the elements $B_{i,j}^{(k)}; i, j = 1, \dots, n$, satisfy the multiplication table for matrix units and the product of any $B_{i,j}^{(k)}$ by any $B_{t,\ell}^{(s)}$ is 0 for $k \neq s$. We note first that if $i, j \neq 1$ then $B_{1,i}^{(k)} B_{1,j}^{(s)} = 0$, since $e_{1,1}e_{1,i}e_{1,1} = 0$ by (7) of Proposition 4.2. Let $i \neq 1$. Then

$$\begin{aligned} B_{1,1}^{(k)} B_{1,i}^{(s)} &= \frac{1}{4} [(e_{1,1}^4 + e_{1,1}^2)e_{1,1}e_{1,i} + (-1)^s (e_{1,1}^4 + e_{1,1}^2)e_{1,1}^2 e_{1,i} \\ &\quad + (-1)^k (e_{1,1}^3 + e_{1,1})e_{1,1}e_{1,i} + (-1)^{k+s} (e_{1,1}^3 + e_{1,1})e_{1,1}^2 e_{1,i}]. \end{aligned}$$

Using (36), (37), (34) of Proposition 4.8 and (24) of Proposition 4.6 implies

$$B_{1,1}^{(k)} B_{1,i}^{(s)} = \frac{1}{4} [2(1 + (-1)^{k+s})e_{1,1}e_{1,i} + 2((-1)^s + (-1)^k)e_{1,1}^2 e_{1,i}] = \delta_{k,s} B_{1,i}^{(k)}.$$

Also,

$$\begin{aligned} B_{1,1}^{(k)} B_{i,1}^{(s)} &= \frac{1}{16} [(e_{1,1}^4 + e_{1,1}^2)e_{i,1}e_{1,1} + (-1)^s (e_{1,1}^4 + e_{1,1}^2)(e_{1,1}^2 e_{i,1} + e_{i,1}) \\ &\quad + (-1)^k (e_{1,1}^3 + e_{1,1})e_{i,1}e_{1,1} + (-1)^{k+s} (e_{1,1}^3 + e_{1,1})(e_{1,1}^2 e_{i,1} + e_{i,1})]. \end{aligned}$$

Using (34), (35), (38) of Proposition 4.8 and (24) of Proposition 4.6 and observing that $e_{1,1}e_{i,1}e_{1,1} = 0$ by (6) of Proposition 4.2 imply

$$\begin{aligned} B_{1,1}^{(k)} B_{i,1}^{(s)} &= \frac{1}{16} [(-1)^s (e_{1,1}^2 e_{i,1} - e_{1,1}^2 e_{i,1} - e_{1,1}^2 e_{i,1} + e_{1,1}^2 e_{i,1}) \\ &\quad + (-1)^{k+s} (e_{1,1}e_{i,1} - e_{1,1}e_{i,1} - e_{1,1}e_{i,1} + e_{1,1}e_{i,1})] = 0. \end{aligned}$$

Next let $i, j \neq 1$. Then

$$\begin{aligned} B_{1,i}^{(k)} B_{j,1}^{(s)} &= \frac{1}{4} [e_{1,1}e_{1,i}e_{j,1}e_{1,1} + (-1)^s e_{1,1}e_{1,i}(e_{1,1}^2 e_{j,1} + e_{j,1}) + (-1)^k e_{1,1}^2 e_{1,i}e_{j,1}e_{1,1} \\ &\quad + (-1)^{k+s} e_{1,1}^2 e_{1,i}(e_{1,1}^2 e_{j,1} + e_{j,1})]. \end{aligned}$$

Using (15) of Proposition 4.4, (8) of Proposition 4.2, (24) and (25) of Proposition 4.6 gives

$$\begin{aligned} B_{1,i}^{(k)} B_{j,1}^{(s)} &= \frac{1}{4} \delta_{j,i} \left[\frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) + \frac{1}{2} (-1)^s (e_{1,1}^3 + e_{1,1}) + \frac{1}{2} (-1)^k (e_{1,1} + e_{1,1}^3) \right. \\ &\quad \left. + \frac{1}{2} (-1)^{k+s} (e_{1,1}^4 + e_{1,1}^2) \right] = \delta_{j,i} \delta_{k,s} B_{1,1}^{(k)}. \end{aligned}$$

Also,

$$\begin{aligned} B_{i,1}^{(k)} B_{1,j}^{(s)} &= \frac{1}{4} [e_{i,1} e_{1,1}^2 e_{1,j} + (-1)^s e_{i,1} e_{1,1}^3 e_{1,j} + (-1)^k (e_{1,1}^2 e_{i,1} e_{1,1} e_{1,j} + e_{i,1} e_{1,1} e_{1,j}) \\ &\quad + (-1)^{k+s} (e_{1,1}^2 e_{i,1} e_{1,1}^2 e_{1,j} + e_{i,1} e_{1,1}^2 e_{1,j})] \\ &= \frac{1}{4} [(1 + (-1)^{k+s}) e_{i,1} e_{1,1}^2 e_{1,j} + ((-1)^s + (-1)^k) e_{i,1} e_{1,1} e_{1,j}], \end{aligned}$$

since $e_{1,1}^3 e_{1,j} = e_{1,1} e_{1,j}$ and $e_{1,1} e_{i,1} e_{1,1} = 0$. Using (15) of Proposition 4.4 and (6) of proposition 4.2 implies

$$\begin{aligned} B_{i,1}^{(k)} B_{1,j}^{(s)} &= \frac{1}{4} [(1 + (-1)^{k+s}) (\delta_{j,i} \frac{1}{2} (e_{1,1}^4 - e_{1,1}^2) + e_{i,1} e_{1,j}) \\ &\quad + ((-1)^s + (-1)^k) (e_{1,j} e_{1,1} e_{i,1} + e_{i,j})] = \delta_{k,s} B_{i,j}^{(k)}. \end{aligned}$$

We have shown that

$$\begin{aligned} B_{1,i}^{(k)} B_{1,j}^{(s)} &= 0, \quad B_{1,1}^{(k)} B_{1,i}^{(s)} = \delta_{k,s} B_{1,i}^{(k)}, \quad B_{1,1}^{(k)} B_{i,1}^{(s)} = 0, \\ B_{1,i}^{(k)} B_{j,1}^{(s)} &= \delta_{k,s} \delta_{j,i} B_{1,1}^{(k)}, \quad B_{i,1}^{(k)} B_{1,j}^{(s)} = \delta_{k,s} B_{i,j}^{(k)}, \end{aligned}$$

for all $i, j \neq 1$. By applying the anti-automorphism η to both sides of the first three products and observing that $B_{1,i}^{(k)} = 4\eta(B_{i,1}^{(k)})$, we obtain

$$B_{j,1}^{(k)} B_{i,1}^{(s)} = 0, \quad B_{i,1}^{(s)} B_{1,1}^{(k)} = \delta_{k,s} B_{i,1}^{(k)}, \quad B_{1,i}^{(s)} B_{1,1}^{(k)} = 0.$$

Now we use the above products to get all the others. For $k, s \in \{0, 1\}$ and $i \neq 1$, we have $B_{1,i}^{(k)} B_{i,1}^{(k)} = B_{1,1}^{(k)}$, hence $B_{1,1}^{(s)} B_{1,i}^{(k)} B_{i,1}^{(k)} = B_{1,1}^{(s)} B_{1,1}^{(k)}$. Thus $B_{1,1}^{(s)} B_{1,1}^{(k)} = \delta_{k,s} B_{1,1}^{(k)}$. We now have $B_{i,q}^{(k)} = B_{i,1}^{(k)} B_{1,q}^{(k)}$ (for all i, q). Hence, $B_{i,q}^{(k)} B_{\ell,t}^{(s)} = B_{i,1}^{(k)} B_{1,q}^{(k)} B_{\ell,1}^{(s)} B_{1,t}^{(s)} = \delta_{k,s} \delta_{q,\ell} B_{i,1}^{(k)} B_{1,t}^{(k)} = \delta_{k,s} \delta_{q,\ell} B_{i,t}^{(k)}$ (for all i, q, ℓ, t). Summarizing

$$(41) \quad B_{i,j}^{(s)} B_{t,\ell}^{(s)} = \delta_{j,t} B_{i,\ell}^{(s)}, \quad B_{i,j}^{(s)} B_{t,\ell}^{(k)} = 0,$$

for all $s, k \in \{0, 1\}$, $s \neq k$ and $i, j, t, \ell = 1, \dots, n$.

We define next the two other sets of $n \times n$ matrix units. For $k \in \{0, 1\}$ and $i, j = 2, \dots, n$, we set

$$\begin{aligned} D_{1,1}^{(k)} &= \frac{1}{4} (e_{1,1}^4 - e_{1,1}^2 + (-1)^k I (e_{1,1} - e_{1,1}^3)), \\ D_{1,i}^{(k)} &= -\frac{1}{2} (e_{1,1} e_{i,1} + (-1)^k I e_{1,1}^2 e_{i,1}), \\ D_{i,1}^{(k)} &= -\frac{1}{2} (e_{1,i} e_{1,1} + (-1)^k I (e_{1,1}^2 e_{1,i} - e_{1,i})), \\ D_{i,i}^{(k)} &= \frac{1}{2} (\frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) - e_{1,i} e_{i,1} - (-1)^k I e_{1,i} e_{1,1} e_{i,1}), \\ D_{i,j}^{(k)} &= -\frac{1}{2} (e_{1,i} e_{j,1} + (-1)^k I e_{1,i} e_{1,1} e_{j,1}); \quad i \neq j, \end{aligned}$$

where $I = \sqrt{-1}$. We wish to show that for each $k \in \{0, 1\}$, the elements $D_{i,j}^{(k)}; i, j = 1, \dots, n$, satisfy the multiplication table for matrix units and the product of any $D_{i,j}^{(k)}$ by any $D_{t,\ell}^{(s)}$ is 0 for $k \neq s$. We note first that if $i, j \neq 1$ then $D_{1,i}^{(k)} D_{1,j}^{(s)} = 0$, since $e_{1,1} e_{i,1} e_{1,1} = 0$. Let $i \neq 1$. Then

$$\begin{aligned} D_{1,1}^{(k)} D_{1,i}^{(s)} &= -\frac{1}{8} [(e_{1,1}^4 - e_{1,1}^2) e_{1,1} e_{i,1} + (-1)^s I (e_{1,1}^4 - e_{1,1}^2) e_{1,1}^2 e_{i,1} \\ &\quad + (-1)^k I (e_{1,1} - e_{1,1}^3) e_{1,1} e_{i,1} - (-1)^{k+s} (e_{1,1} - e_{1,1}^3) e_{1,1}^2 e_{i,1}]. \end{aligned}$$

Using (35), (38), (34) of Proposition 4.8 and (24) of Proposition 4.6 gives

$$\begin{aligned} D_{1,1}^{(k)} D_{1,i}^{(s)} &= -\frac{1}{8} [2e_{1,1}e_{i,1} + 2(-1)^s I e_{1,1}^2 e_{i,1} + 2(-1)^k I e_{1,1}^2 e_{i,1} + 2(-1)^{k+s} e_{1,1}e_{i,1}] \\ &= -\frac{1}{4} [(1 + (-1)^{k+s}) e_{1,1}e_{i,1} + ((-1)^s + (-1)^k) I e_{1,1}^2 e_{i,1}] = \delta_{k,s} D_{1,i}^{(k)}. \end{aligned}$$

Also,

$$\begin{aligned} D_{1,1}^{(k)} D_{i,1}^{(s)} &= -\frac{1}{8} [e_{1,1}^4 e_{1,i} e_{1,1} + (-1)^s I e_{1,1}^4 (e_{1,1}^2 e_{1,i} - e_{1,i}) - e_{1,1}^2 e_{1,i} e_{1,1} \\ &\quad - (-1)^s I e_{1,1}^2 (e_{1,1}^2 e_{1,i} - e_{1,i}) + (-1)^k I (e_{1,1} - e_{1,1}^3) e_{1,i} e_{1,1} \\ &\quad - (-1)^{k+s} (e_{1,1} - e_{1,1}^3) (e_{1,1}^2 e_{1,i} - e_{1,i})]. \end{aligned}$$

Using (34), (36), (37) of Proposition 4.8 and (24) of Proposition 4.6 and observing that $e_{1,1}e_{1,i}e_{1,1} = 0$ imply

$$\begin{aligned} D_{1,1}^{(k)} D_{i,1}^{(s)} &= -\frac{1}{8} [(-1)^s I (e_{1,1}^2 e_{1,i} - e_{1,1}^2 e_{1,i}) - (-1)^s I (e_{1,1}^2 e_{1,i} - e_{1,1}^2 e_{1,i}) \\ &\quad - (-1)^{k+s} (e_{1,1}e_{1,i} - e_{1,1}e_{1,i} - e_{1,1}e_{1,i} + e_{1,1}e_{1,i})] = 0. \end{aligned}$$

Next let $i, j \neq 1$. Then

$$\begin{aligned} D_{1,i}^{(k)} D_{j,1}^{(s)} &= \frac{1}{4} [e_{1,1}e_{i,1}e_{1,j}e_{1,1} + (-1)^s I (e_{1,1}e_{i,1}e_{1,1}^2 e_{1,j} - e_{1,1}e_{i,1}e_{1,j}) \\ &\quad + (-1)^k I e_{1,1}^2 e_{i,1}e_{1,j}e_{1,1} - (-1)^{k+s} e_{1,1}^2 e_{i,1} (e_{1,1}^2 e_{1,j} - e_{1,j})]. \end{aligned}$$

Using (20) of Proposition 4.5, (6) of Proposition 4.2, (30) and (31) of Proposition 4.7 gives

$$\begin{aligned} D_{1,i}^{(k)} D_{j,1}^{(s)} &= \frac{1}{4} [\delta_{i,j} \frac{1}{2} (e_{1,1}^4 - e_{1,1}^2) - \delta_{i,j} (-1)^s I \frac{1}{2} (e_{1,1}^3 - e_{1,1}) + \delta_{i,j} (-1)^k \frac{1}{2} I (e_{1,1} - e_{1,1}^3) \\ &\quad + \delta_{i,j} (-1)^{k+s} \frac{1}{2} (e_{1,1}^4 - e_{1,1}^2)] \\ &= \frac{1}{4} \delta_{i,j} [\frac{1}{2} (1 + (-1)^{k+s}) (e_{1,1}^4 - e_{1,1}^2) + \frac{1}{2} ((-1)^s + (-1)^k) I (e_{1,1} - e_{1,1}^3)] \\ &= \delta_{s,k} \delta_{i,j} D_{1,1}^{(k)}. \end{aligned}$$

Also,

$$\begin{aligned} D_{i,1}^{(k)} D_{1,\ell}^{(s)} &= \frac{1}{4} [e_{1,i}e_{1,1}^2 e_{\ell,1} + (-1)^s I e_{1,i}e_{1,1}^3 e_{\ell,1} + (-1)^k I (e_{1,1}^2 e_{1,i}e_{1,1}e_{\ell,1} - e_{1,i}e_{1,1}e_{\ell,1}) \\ &\quad - (-1)^{k+s} (e_{1,1}^2 e_{1,i}e_{1,1}^2 e_{\ell,1} - e_{1,i}e_{1,1}^2 e_{\ell,1})]. \end{aligned}$$

Using (11) of Proposition 4.3 and (23) of Proposition 4.5 implies

$$\begin{aligned} D_{i,1}^{(k)} D_{1,\ell}^{(s)} &= \frac{1}{4} [(1 + (-1)^{k+s}) (\delta_{i,\ell} \frac{1}{2} (e_{1,1}^4 + e_{1,1}^2) - e_{1,i}e_{\ell,1}) \\ &\quad - ((-1)^s + (-1)^k) I e_{1,i}e_{1,1}e_{\ell,1}] = \delta_{k,s} D_{i,\ell}^{(k)}. \end{aligned}$$

The other products can be obtained by using the argument at the end of the proof of the first two sets of $n \times n$ matrix units. Summarizing

$$(42) \quad D_{i,j}^{(s)} D_{k,\ell}^{(s)} = \delta_{j,k} D_{i,\ell}^{(s)}, \quad D_{i,j}^{(s)} D_{k,\ell}^{(t)} = 0,$$

for all $s, t \in \{0, 1\}$, $s \neq t$ and $i, j, k, \ell = 1, \dots, n$.

We wish to prove now that the product of any $D_{i,j}^{(k)}$ by any $B_{m,n}^{(s)}$ is 0. Clearly $D_{1,i}^{(k)} B_{\ell,1}^{(s)} = 0$ and $D_{1,i}^{(k)} B_{1,\ell}^{(s)} = 0$ ($i, \ell \neq 1$), since $e_{i,1}e_{\ell,1} = 0$ and $e_{1,1}e_{i,1}e_{1,1} = 0$.

Let $\ell \neq 1$. Then

$$\begin{aligned} D_{1,1}^{(k)} B_{1,\ell}^{(s)} &= \frac{1}{4} [(e_{1,1}^4 - e_{1,1}^2) e_{1,1} e_{1,\ell} + (-1)^s (e_{1,1}^4 - e_{1,1}^2) e_{1,1}^2 e_{1,\ell} \\ &\quad + (-1)^k \mathbf{I} (e_{1,1} - e_{1,1}^3) e_{1,1} e_{1,\ell} + (-1)^{k+s} \mathbf{I} (e_{1,1} - e_{1,1}^3) e_{1,1}^2 e_{1,\ell}] \\ &= \frac{1}{4} [(e_{1,1} e_{1,\ell} - e_{1,1} e_{1,\ell}) + (-1)^s (e_{1,1}^2 e_{1,\ell} - e_{1,1}^2 e_{1,\ell}) \\ &\quad + (-1)^k \mathbf{I} (e_{1,1}^2 - e_{1,1}^2) e_{1,\ell} + (-1)^{k+s} \mathbf{I} (e_{1,1} - e_{1,1}) e_{1,\ell}] = 0, \end{aligned}$$

using (36), (37), (34) of Proposition 4.8 and (24) of Proposition 4.6. Also,

$$\begin{aligned} D_{1,1}^{(k)} B_{\ell,1}^{(s)} &= \frac{1}{16} [(e_{1,1}^4 - e_{1,1}^2) e_{\ell,1} e_{1,1} + (-1)^s (e_{1,1}^4 - e_{1,1}^2) (e_{1,1}^2 e_{\ell,1} + e_{\ell,1}) \\ &\quad + (-1)^k \mathbf{I} (e_{1,1} - e_{1,1}^3) e_{\ell,1} e_{1,1} + (-1)^{k+s} \mathbf{I} (e_{1,1} - e_{1,1}^3) (e_{1,1}^2 e_{\ell,1} + e_{\ell,1})] \\ &= \frac{1}{16} [(-1)^s (e_{1,1}^2 e_{\ell,1} - e_{1,1}^2 e_{\ell,1} - (-e_{1,1}^2 e_{\ell,1} + e_{1,1}^2 e_{\ell,1})) \\ &\quad + (-1)^{k+s} \mathbf{I} (-e_{1,1} e_{\ell,1} + e_{1,1} e_{\ell,1} - e_{1,1} e_{\ell,1} + e_{1,1} e_{\ell,1})] = 0, \end{aligned}$$

using (38), (34), (35) of Proposition 4.8 and (24) of Proposition 4.6. We have shown that

$$(43) \quad D_{1,i}^{(k)} B_{\ell,1}^{(s)} = 0, \quad D_{1,i}^{(k)} B_{1,\ell}^{(s)} = 0, \quad D_{1,1}^{(k)} B_{1,\ell}^{(s)} = 0, \quad D_{1,1}^{(k)} B_{\ell,1}^{(s)} = 0,$$

for all $i, \ell \neq 1$ and $k, s \in \{0, 1\}$. Let $\ell \neq 1$, then $D_{1,i}^{(k)} B_{1,1}^{(s)} = D_{1,i}^{(k)} B_{1,\ell}^{(s)} B_{\ell,1}^{(s)} = 0$ (for all i), using (41) and (43). Combining this result with the first and the last equations of (43) gives $D_{1,i}^{(k)} B_{j,1}^{(s)} = 0$ (for all i, j). By (41) and (42), $D_{i,j}^{(k)} B_{t,\ell}^{(s)} = D_{i,1}^{(k)} D_{1,j}^{(k)} B_{t,1}^{(s)} B_{1,\ell}^{(s)}$ (for all i, j, t, ℓ). Hence, $D_{i,j}^{(k)} B_{t,\ell}^{(s)} = 0$ (for all i, j, t, ℓ). By using the anti-automorphism η , we can show that $B_{t,\ell}^{(s)} D_{i,j}^{(k)} = 0$ (for all i, j, t, ℓ). Summarizing

$$(44) \quad D_{i,j}^{(k)} B_{t,\ell}^{(s)} = 0 = B_{t,\ell}^{(s)} D_{i,j}^{(k)} \quad \text{for all } i, j, t, \ell = 1, \dots, n, \quad s, k \in \{0, 1\}.$$

Finally, we define the set of 1×1 matrix unit. We set,

$$(45) \quad A_{1,1} = \sum_{i=2}^n e_{1,i} e_{i,1} - \sum_{i=2}^n e_{i,1} e_{1,i} - n e_{1,1}^4 + 1.$$

We wish to show that $A_{1,1}^2 = A_{1,1}$ and the products of $A_{1,1}$ by any $B_{i,j}^{(k)}$ and $D_{i,j}^{(k)}$ are 0. We observe that

$$\begin{aligned} (46) \quad &\sum_{k=0}^1 (B_{1,1}^{(k)} + D_{1,1}^{(k)}) + \sum_{i=2}^n \sum_{k=0}^1 B_{i,i}^{(k)} + \sum_{i=2}^n \sum_{k=0}^1 D_{i,i}^{(k)} \\ &= e_{1,1}^4 + \sum_{i=2}^n (\frac{1}{2}(e_{1,1}^4 - e_{1,1}^2) + e_{i,1} e_{1,i}) + \sum_{i=2}^n (\frac{1}{2}(e_{1,1}^4 + e_{1,1}^2) - e_{1,i} e_{i,1}) \\ &= e_{1,1}^4 + \frac{1}{2}(n-1)(e_{1,1}^4 - e_{1,1}^2) + \sum_{i=2}^n e_{i,1} e_{1,i} + \frac{1}{2}(n-1)(e_{1,1}^4 + e_{1,1}^2) - \sum_{i=2}^n e_{1,i} e_{i,1} \\ &= n e_{1,1}^4 + \sum_{i=2}^n e_{i,1} e_{1,i} - \sum_{i=2}^n e_{1,i} e_{i,1}. \end{aligned}$$

Using (46) in (45) gives

$$(47) \quad A_{1,1} = 1 - \left(\sum_{k=0}^1 (B_{1,1}^{(k)} + D_{1,1}^{(k)}) + \sum_{i=2}^n \sum_{k=0}^1 B_{i,i}^{(k)} + \sum_{i=2}^n \sum_{k=0}^1 D_{i,i}^{(k)} \right).$$

Multiply (47) by $B_{\ell,m}^{(k)}$ from the right and use the relations of (41), (42) and (44) (of the present proof), we obtain

$$A_{1,1} B_{\ell,m}^{(k)} = B_{\ell,m}^{(k)} - B_{\ell,m}^{(k)} = 0.$$

Similarly, we can show that $B_{\ell,m}^{(k)} A_{1,1} = 0$ and $A_{1,1} D_{\ell,m}^{(k)} = 0 = D_{\ell,m}^{(k)} A_{1,1}$. To show $A_{1,1}^2 = A_{1,1}$, we multiply (47) by $A_{1,1}$ and use the last discussion.

Now let $\Phi_n^{(k)}$ (resp. $\Psi_n^{(k)}$ and τ_1) denote the subspace of \mathfrak{A} generated by the $B_{i,j}^{(k)}$ (resp. $D_{i,j}^{(k)}$ and $A_{1,1}$), $k \in \{0, 1\}$. Our discussion shows that $\Phi_n^{(k)}$ (resp. $\Psi_n^{(k)}$ and τ_1) is a subalgebra of \mathfrak{A} and isomorphic to $M_{n,n}$ (resp. $M_{n,n}$ and $M_{1,1}$), $\Phi_n^{(k)} \Phi_n^{(s)} = 0 = \Phi_n^{(s)} \Phi_n^{(k)}$, $\Psi_n^{(k)} \Psi_n^{(s)} = 0 = \Psi_n^{(s)} \Psi_n^{(k)}$ ($k \neq s$), $\Phi_n^{(k)} \Psi_n^{(s)} = 0 = \Psi_n^{(s)} \Phi_n^{(k)}$, $\Phi_n^{(s)} \tau_1 = 0 = \tau_1 \Phi_n^{(s)}$ and $\Psi_n^{(s)} \tau_1 = 0 = \tau_1 \Psi_n^{(s)}$. By (47) and the definitions of $B_{i,j}^{(k)}$ and $D_{i,j}^{(k)}$, we have

$$(48) \quad \begin{aligned} 1 &= A_{1,1} + \sum_{i=1}^n B_{i,i}^{(0)} + \sum_{i=1}^n B_{i,i}^{(1)} + \sum_{i=1}^n D_{i,i}^{(0)} + \sum_{i=1}^n D_{i,i}^{(1)}, \\ e_{1,1} &= B_{1,1}^{(0)} - B_{1,1}^{(1)} - I D_{1,1}^{(0)} + I D_{1,1}^{(1)}, \\ e_{i,j} &= B_{i,j}^{(0)} - B_{i,j}^{(1)} - I D_{j,i}^{(0)} + I D_{j,i}^{(1)}; \quad i, j \neq 1, i \neq j, \\ e_{i,i} &= B_{i,i}^{(0)} - B_{i,i}^{(1)} - I D_{i,i}^{(0)} + I D_{i,i}^{(1)}; \quad i \neq 1, \\ e_{1,i} &= \frac{1}{2} B_{1,i}^{(0)} - \frac{1}{2} B_{1,i}^{(1)} - I D_{i,1}^{(0)} + I D_{i,1}^{(1)}; \quad i \neq 1, \\ e_{i,1} &= 2 B_{i,1}^{(0)} - 2 B_{i,1}^{(1)} - I D_{1,i}^{(0)} + I D_{1,i}^{(1)}; \quad i \neq 1. \end{aligned}$$

Thus all the $1, e_{i,j} \in \tau_1 \oplus \Phi_n^{(0)} \oplus \Phi_n^{(1)} \oplus \Psi_n^{(0)} \oplus \Psi_n^{(1)}$. Hence $\mathfrak{A} = \tau_1 \oplus \Phi_n^{(0)} \oplus \Phi_n^{(1)} \oplus \Psi_n^{(0)} \oplus \Psi_n^{(1)}$. \square

Remark 6.2. The equations (48) (of the last proof) describe all inequivalent irreducible representations of the anti-Jordan triple system \mathfrak{J} .

Corollary 6.3. *The universal enveloping algebra of the simple anti-Jordan triple system of all $n \times n$ matrices over an algebraically closed field is semisimple.*

The next example shows that the universal enveloping algebra is not necessary to be finite-dimensional

Example 6.4. Consider the 2-dimensional anti-Jordan triple system S with basis $\mathcal{B} = \{a = e_{1,2}, b = e_{2,1}\}$ of matrix units and triple product given by $\langle a, b, c \rangle = abc - cba$. It is easy to check that the multiplication table of S is zero. The universal enveloping algebra is associative algebra with relations: $b^2 a = ab^2$ and $ba^2 = a^2 b$, which is the down-up algebra $A(0, 1, 0)$ (see [2]).

To conclude the paper, we formulate the following conjecture.

Conjecture. *If the universal enveloping algebra of a simple finite-dimensional anti-Jordan triple system is finite-dimensional, then it is semisimple.*

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